A Generalized Approach to Indeterminacy in Linear Rational Expectations Models*

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23rd April 2020

Abstract

We propose a novel approach to deal with the problem of indeterminacy in Linear Rational Expectations models. The method consists of augmenting the original state space with a set of auxiliary exogenous equations to provide the adequate number of explosive roots in presence of indeterminacy. The solution in this expanded state space, if it exists, is always determinate, and is identical to the indeterminate solution of the original model. The proposed approach accommodates determinacy and any degree of indeterminacy, and it can be implemented even when the boundaries of the determinacy region are unknown. Thus, the researcher can estimate the model using standard software packages without restricting the estimates to the determinacy region. We combine our solution method with a novel hybrid Metropolis-Hastings algorithm to estimate the New-Keynesian model with rational bubbles by Galí (forthcoming) over the period 1982:Q4 until 2007:Q3. We find that the data support the presence of two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component.

JEL classification: C19, C51, C62, C63.

Keywords: Indeterminacy, General Equilibrium, Solution method, Bayesian methods.

*The views expressed in this paper are those of the authors and do not reflect the views of the Federal Reserve Board or the Federal Reserve System. We thank Jonas Arias, Jess Benhabib, Roger Farmer, Francois Geerolf, Frank Schorfheide, and all participants at UCLA seminars, NBER Multiple Equilibria and Financial Crises Conference, CEPR-IMFS New Methods for Macroeconomic Modeling, Model Comparison and Policy Analysis Conference, Federal Reserve Bank of St. Louis, Society of Economic Dynamics, 12th Dynare Conference, 2017 NBER-NSF conference on Bayesian Inference in Econometrics and Statistics.

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1 Introduction

Sunspot shocks and multiple equilibria have been at the center of economic thinking at least since the seminal work of Cass and Shell (1983), Farmer and Guo (1994), and Farmer and Guo (1995). The zero lower bound has brought renovated interest to the problem of indeterminacy (Aruoba et al., 2018). Furthermore, in many of the Linear Rational Expectation (LRE) models used to study the properties of the macroeconomy the possibility of multiple equilibria arises for some parameter values, but not for others. This paper proposes a novel approach to solve LRE models that easily accommodates both the case of determinacy and indeterminacy. As a result, the proposed methodology can be used to easily solve and estimate a LRE model that could potentially be characterized by multiplicity of equilibria. Our approach is implementable even when the analytic conditions for determinacy or the degrees of indeterminacy are unknown. Importantly, the proposed method can be easily implemented to study indeterminacy in standard software packages, such as Dynare (Adjemian et al., 2011) and Sims' (2001) code Gensys.

To understand how our approach works, it is useful to recall the conditions for determinacy as stated by Blanchard and Kahn (1980). Indeterminacy arises when the parameter values are such that the number of explosive roots is smaller than the number of non-predetermined variables. The key idea behind our methodology consists of augmenting the original model by appending additional autoregressive processes that can be used to provide the missing explosive roots. The innovations of these exogenous processes are assumed to be linear combinations of a subset of the forecast errors associated with the expectational variables of the model and a newly defined vector of sunspot shocks. When the Blanchard-Kahn condition for determinacy is satisfied, all the roots of the auxiliary autoregressive processes are assumed to be within the unit circle and the auxiliary process is irrelevant for the dynamics of the model. The law of motion for the endogenous variables is in this case equivalent to the solution obtained using standard solution algorithms (King and Watson, 1998; Klein, 2000; Sims, 2001). When the model is indeterminate, the appropriate number of appended autoregressive processes is assumed to be explosive. For example, if there are two degrees of indeterminacy, two of the auxiliary processes are assumed to be explosive. The solution that we obtain for the endogenous variables is equivalent to the one obtained with the methodology of Lubik and Schorfheide (2003) or, equivalently, Farmer et al. (2015).

Our methodology simplifies the common approach used to deal with indeterminacy. The common procedure requires the researcher to solve the model differently depending on the area of the parameter space that is being studied. Under indeterminacy, existing methods require to construct the solution ex-post following the seminal contribution of Lubik and Schorfheide (2003) or to rewrite the model based on the existing degree of indeterminacy (Farmer et al., 2015). In itself, this is not an insurmountable task, but it implies that the researcher cannot simply use standard solution methods or software packages. What is more, if the researcher is interested in a structural estimation of the model, she would need to write the estimation codes and not just the solution codes. Our proposed method only requires the researcher to augment the original system of equations to reflect the maximum degree of indeterminacy and can therefore be used with no modification of the standard solution approaches. Thus, our methodology can be used with standard estimation packages such as Dynare.

Our method can also be combined with more sophisticated Bayesian techniques to facilitate the transition between the determinacy and indeterminacy regions of the parameter space. To elucidate this point, we propose an hybrid Metropolis-Hastings algorithm that builds on a specific example developed in An and Schorfheide (2007) following Giordani et al. (2010). The algorithm combines the standard Metropolis-Hastings random walk algorithm with a Markov Chain Monte Carlo (MCMC) algorithm in which the proposal distribution is based on a mixture of normals centered on the different posterior modes. This algorithm guarantees that the chain quickly moves to the region of the parameter space with the highest posterior and allows to visit local peaks with more frequency. We adopt our proposed solution method and hybrid Metropolis-Hasting algorithm to estimate the small-scale New-Keynesian (NK) model of Galí (forthcoming) using Bayesian techniques on U.S. data over the period 1982:Q4 until 2007:Q3. Galí's model extends a conventional NK model to allow for the existence of rational bubbles. An interesting aspect of the model is that it displays up to two degrees of indeterminacy for realistic parameter values. We find that the data support the version of the model with two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component. Importantly, we show that the combination of our method with the hybrid algorithm ensures a more efficient estimation and faster transition to the region of the parameter space with the best model fit relative to a standard Metropolis-Hastings random walk algorithm. In Subsection 6.2, we reconsider the NK model of Lubik and Schorfheide (2004) to show that the algorithm also speeds up the convergence rate in presence of multiple peaks in the posterior.

Our work is related to the vast literature that studies the role of indeterminacy in explaining the evolution of the macroeconomy. Prominent examples in the monetary policy literature include the work of Clarida et al. (2000) and Kerr and King (1996), that study the possibility of multiple equilibria as a result of violations of the Taylor Principle in NK models. Applying the methods developed in Lubik and Schorfheide (2003) to the canonical NK model, Lubik and Schorfheide (2004) test for indeterminacy in U.S. monetary policy. Using a calibrated small-scale model, Coibon and Gorodnichenko (2011) find that the reduction of the target inflation rate in the United States also played a key role in explaining the Great Moderation, and Arias et al. (forthcoming) support this finding in the context of a medium-scale model *à la* Christiano et al. (2005). More recently, Aruoba et al. (2018) study inflation dynamics at the Zero Lower Bound (ZLB) and during an exit from the ZLB.

The paper closest to our is Farmer et al. (2015). As explained above, the main difference between the two approaches is that our method accommodates the case of both determinacy and indeterminacy while considering the same augmented system of equations. Instead, the method proposed by Farmer et al. (2015) requires us to rewrite the model based on the existing degree of indeterminacy.

With respect to Lubik and Schorfheide (2003), our theoretical results show the characterization of the full set of equilibria under indeterminacy is equivalent between the two methods. A novelty of our approach is to provide a unified methodology to study determinacy and indeterminacy of different degrees.[1] Lubik and Schorfheide (2004) propose a baseline solution that minimizes the distance between the impulse responses of the model under indeterminacy and determinacy evaluated at the boundary of the region of determinacy. Our theoretical results show the equivalence between our methods and therefore the possibility of mapping their baseline solution into our representation. However, in the context of our solution method, it is more natural to think about the baseline solution as the one that restricts to zero the *contemporaneous* impact of the fundamental shocks on the variables whose sunspot shocks are considered as drivers of the economy. To understand why, consider that under indeterminacy, in our approach, the expectational variables whose forecast errors are included in the explosive auxiliary processes *always* become predetermined and their contemporaneous changes are *only* a function of the respective sunspot shocks. Thus, fundamental shocks can affect these variables only through their effect on sunspot shocks. Therefore, our method naturally suggests a baseline solution that sets the correlations of the fundamental disturbances with the sunspot shocks to zero. Such identification can be equivalently thought as an assumption that fundamental shocks do not have a contemporaneous impact on those variables. This identification strategy is reminiscent of the zero restrictions often used in the Structural VAR (SVAR) literature. Importantly, this choice for the baseline solution does not impose a constraint on a researcher who wants to consider alternative solutions. Instead, the baseline solution is just meant to provide an intuitive and simple-to-implement benchmark.

The remainder of the paper is organized as follows. Section 2 builds the intuition by using a univariate example in the spirit of Lubik and Schorfheide (2004). In Section 3, we present the methodology and show that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). In addition, we also provide a detailed explanation

[1] Ascari et al. (2019) allow for temporarily unstable paths, while we require all solutions to be stationary, in line with previous contributions in the literature.

of how to construct the baseline solution in our method. In Section 4, we describe the hybrid algorithm that we propose to facilitate the estimation of models with different degrees of indeterminacy. In Section 5, we apply our theoretical results to estimate the NK model with rational bubbles of Galí (forthcoming) using Bayesian techniques and the proposed hybrid algorithm. Section 6 provides practical suggestions for the implementation of our method. First, we highlight the advantages of adopting more sophisticated estimation techniques such as the proposed hybrid algorithm to estimate models with indeterminacy of different degrees. Second, we provide tips for the adoption of our method using standard software packages such as Dynare. We present our conclusions in Section 7.

2 Building the intuition

Before presenting the theoretical results of the paper, this section builds the intuition behind our approach by considering a univariate example similar to the one proposed in Lubik and Schorfheide (2004). While Subsection 2.1 explains our approach from an analytical perspective, Subsection 2.2 addresses questions which could arise at the time of its practical implementation.

2.1 A useful example

Consider a classical monetary model characterized by the Fisher equation

$$i_t = E_t(\pi_{t+1}) + r_t, \tag{1}$$

and the simple Taylor rule

$$i_t = \phi_\pi \pi_t, \tag{2}$$

where i_t denotes the nominal interest rate, π_t represents the inflation rate, and $\phi_\pi > 0$ is a parameter controlling the response of the nominal interest to inflation. We assume that the real interest rate, r_t,is given and described by a mean-zero Gaussian i.i.d. shock.[2] To properly specify the model, we also define the one-step ahead forecast error associated with the expectational variable, π_t, as

$$\eta_t \equiv \pi_t - E_{t-1}(\pi_t). \tag{3}$$

[2]In the classical monetary model, the real interest rate results from the equilibrium in labor and goods market, and it depends on the technology shocks. We are considering an exogenous process for the technology shocks, and therefore we take the process for the real interest rate as given.

Combining (1) and (2), we obtain the univariate model

$$E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t. \tag{4}$$

Any solution to (4) satisfies

$$\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \eta_t. \tag{5}$$

First, we consider the case $\phi_\pi > 1$. Solving (5) forward and recalling the assumptions on r_t, it is clear that this case is associated with the determinate solution

$$\pi_t = \frac{1}{\phi_\pi} r_t. \tag{6}$$

The strong response of the monetary authority to changes in inflation ($\phi_\pi > 1$) guarantees that inflation is pinned down as a function of the exogenous real interest rate r_t. From a technical perspective, when $\phi_\pi > 1$ the Blanchard-Kahn condition for uniqueness of a solution is satisfied: The number of explosive roots matches the number of expectational variables, that in this univariate case is one.

The second case corresponds to $\phi_\pi \leq 1$. The solution corresponds to *any* process that takes the form in (5). Such solution also holds under determinacy, but in that case the central bank's behavior induces restrictions on the expectation error η_t as a function of the exogenous shock, r_t. Instead, when the monetary authority does not respond aggressively enough to changes in inflation ($\phi_\pi \leq 1$), there are multiple solutions for the inflation rate, π_t, each indexed by the expectations that the representative agent holds about future inflation, η_t. Equivalently, the solution to the univariate model is indeterminate: The Blanchard-Kahn solution is not satisfied as there is no explosive root to match the number of expectational variables.[3]

The simple model considered here can be solved pencil and paper. However, when considering richer models with multiple endogenous variables, indeterminacy represents a challenge from a methodological and computational perspective. Standard software packages such as Dynare do not allow for indeterminacy. Of course, a researcher could in principle code an estimation algorithm herself, following the methods outlined in Lubik and Schorfheide (2004). However, this approach requires a substantial amount of time and technical skills. The researcher would need to write a code that not only finds the solution, but also implements the estimation algorithm. Hence, the result is that in practice most of the papers simply rule out the possibility of indeterminacy, even if the model at hand could in principle allow for such a feature.

The problem that a researcher faces when solving a LRE model under indeterminacy using

[3] To ensure boudedness, the indeterminate solution requires the forecast error, η_t, to be any covariance-stationary martingale difference process.

standard solution algorithms can be easily understood based on the example provided above. Under determinacy, the model already has a sufficient number of unstable roots to match the number of expectational variables. However, under indeterminacy, the model is missing one explosive root. Thus, we propose to augment the original state space of the model by appending an independent process which could be either stable or unstable.

The key insight consists of choosing this auxiliary processes in a way to deliver the correct solution. When the original model is determinate, the auxiliary process must be stationary so that also the augmented representation satisfies the Blanchard-Kahn condition. In this case, the auxiliary process represents a separate block that does not affect the law of motion of the model variables. When the model is indeterminate, the additional process should however be explosive so that the Blanchard-Kahn condition is satisfied for the augmented system, even if not for the original model. By choosing the auxiliary process in the appropriate way, the solution under determinacy in this expanded state space corresponds to the solution under indeterminacy under the original state space. In what follows, we apply this intuition to the example considered above. In Section 3, we show that the approach can be easily extended to richer models to accommodate any degree of indeterminacy.

Our methodology proposes to solve an augmented system of equations which can be dealt with by using standard solution algorithms such as Sims (2001) under both determinacy and indeterminacy. Consider the following augmented system

$$\begin{cases} E_t(\pi_{t+1}) = \phi_\pi \pi_t - r_t, \\ \omega_t = \left(\frac{1}{\alpha}\right)\omega_{t-1} - \nu_t + \eta_t, \end{cases} \tag{7}$$

where ω_t is an independent autoregressive process, $\alpha \in [0, 2]$ and ν_t is a newly defined mean-zero sunspot shock with standard deviation σ_ν.

Table 1 summarizes the intuition behind our approach. When the original LRE model in (4) is determinate, $\phi_\pi > 1$, the Blanchard-Kahn condition for the augmented representation in (7) is satisfied when $1/\alpha < 1$. Indeed, for $\phi_\pi > 1$ the original model has the same number of unstable roots as the number of expectational variables. Our methodology thus suggests to append a stable autoregressive process such that $1/\alpha < 1$. In this case, the method of Sims (2001) delivers the same solution for the endogenous variable π_t as in equation (6). Importantly, ω_t is an independent autoregressive process, and its dynamics do not impact the endogenous variable π_t.

Considering the case of indeterminacy (i.e. $\phi_\pi \leq 1$), the original model has one expectational variable, but no unstable root, thus violating the Blanchard-Kahn condition. By appending an explosive autoregressive process, the augmented representation that we propose satisfies the Blanchard-Kahn condition and delivers the same solution as the one resulting from the meth-

Table 1: Blanchard-Kahn condition in the augmented representation

Determinacy $\phi_\pi > 1$ in original model (4)	Unstable roots	B-K condition in augmented model (7)	Solution
$\frac{1}{\alpha} < 1$	1	Satisfied	$\begin{cases} \pi_t = \frac{1}{\phi_\pi} r_t \\ \omega_t = \alpha\omega_{t-1} - \nu_t + \eta_t \\ \eta_t = \frac{1}{\phi_\pi} r_t \end{cases}$
$\frac{1}{\alpha} > 1$	2	Not satisfied	-
Indeterminacy $\phi_\pi \leq 1$ in original model (4)			
$\frac{1}{\alpha} < 1$	0	Not satisfied	-
$\frac{1}{\alpha} > 1$	1	Satisfied	$\begin{cases} \pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \nu_t \\ \omega_t = 0 \\ \eta_t = \nu_t \end{cases}$

The table reports the regions of the parameter space for which the Blanchard-Kahn condition in the augmented representation is satisfied, even when the original model is indeterminate.

odology of Lubik and Schorfheide (2003) or Farmer et al. (2015) described by equation (5). Moreover, stability imposes conditions such that ω_t is always equal to zero at any time t, thus requiring to impose ω_0 equal to zero and $\eta_t = \nu_t$. Importantly, even in this case, the solution for the endogenous variable does not depend on the appended autoregressive process.

Summarizing, the choice of the coefficient $1/\alpha$ should be made as follows. For values of ϕ_π greater than 1, the Blanchard-Kahn condition for the augmented representation is satisfied for values of α greater than 1. Conversely, under indeterminacy (i.e. $\phi_\pi \leq 1$) the condition is satisfied when α is smaller than 1. The choice of parametrizing the auxiliary process with $1/\alpha$ instead of α induces a positive correlation between ϕ_π and α that facilitates the implementation of our method when estimating a model.

Finally, note that under both determinacy and indeterminacy, the exact value of $1/\alpha$ is irrelevant for the law of motion of π_t. Under determinacy, the auxiliary process ω_t is stationary, but its evolution does not affect the law of motion of the model variables. Under indeterminacy, ω_t is always equal to zero. Thus, the introduction of the auxiliary process does not affect the properties of the solution in the two cases. However, this process serves two important purposes: It provides the correct number of explosive roots under indeterminacy and creates a mapping between the sunspot shock and the expectation errors. As we will see in Section 3, this result can be generalized and applies to more complicated models with potentially multiple degrees of indeterminacy.

2.2 Choosing α

Before presenting detailed suggestions for the practical implementation of our method in Section 6, it is useful to provide the intuition for the choice of the parameter α in the context of the simple model presented above. First of all, from the discussion above, it should be clear that what matters is only if this parameter is smaller or larger than 1. Its exact value does not affect the solution for π_t. Thus, if a researcher wants to solve the model only under indeterminacy (determinacy), it can simply fix the parameter to a value smaller (larger) than 1. In this way, standard solution algorithms proceed to solve the model in the augmented state space only when the model under the original state space is characterized by indeterminacy (determinacy).

However, a researcher might want to allow for both determinacy and indeterminacy when solving the model. We consider the following two cases: (1) The analytic conditions defining the region of determinacy are known; (2) The analytic conditions defining the region of determinacy are unknown. We consider the two cases separately.

We first consider the case in which the researcher is able to analytically derive the condition which defines when the model is determinate or indeterminate. For the example considered in this section, this case corresponds to knowing that when $\phi_\pi \leq 1$ the model in (4) is indeterminate. We thus suggest to write the parameter α as a function of the parameter ϕ_π so that the augmented representation in (7) always satisfies the Blanchard-Kahn condition. In this example, we set $\alpha \equiv \phi_\pi$. When the original model is determinate ($\phi_\pi > 1$), the appended autoregressive process is stationary because $1/\alpha < 1$. If the original model is indeterminate ($\phi_\pi \leq 1$), the coefficient $1/\alpha$ is greater than 1 and the appended process is therefore explosive. Hence, when the region of determinacy is known, the researcher can easily choose α such that the augmented representation always delivers a solution under both determinacy and indeterminacy. Note that in this case α is a transformation of ϕ_π and effectively no auxiliary extra parameters are introduced.

There are however instances in which the researcher does not know the exact properties of the determinacy region. In this case, the researcher can start with an arbitrary value of α for a given sets of parameters θ. Suppose that the researcher starts with a value less than 1 and finds that the model is indeterminate for the given set of parameters θ. Then, the researcher can just change α to a value larger than 1, for example $\alpha' = 1/\alpha$. A similar logic applies to the case with multiple degrees of indeterminacy that we discuss below: If the solution algorithm returns a solution with m degrees of indeterminacy, m explosive auxiliary processes are necessary.[4]

[4] Here, α can be thought as an unkown transformation of the structural parameters or as an additional parameter. While this distinction does not make any difference for the solution of the model, it does for the estimation of the model, as we discuss in Subsection 6.2.

3 Methodology

We now present the main contribution of the paper generalizing the intuition provided above to a multivariate model with potentially multiple degrees of indeterminacy. Given the general class of LRE models described in Sims (2001), this paper proposes an augmented representation which embeds the solution for the model under both determinacy and indeterminacy. In particular, the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, and under indeterminacy to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) or equivalently Farmer et al. (2015). In the following, we generalize the intuition built in the previous section. Consider the following LRE model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{8}$$

where $X_t \in R^k$ is a vector of endogenous variables, $\varepsilon_t \in R^\ell$ is a vector of exogenous shocks, $\eta_t \in R^p$ collects the p one-step ahead forecast errors for the expectational variables of the system and $\theta \equiv vec(\Gamma_0, \Gamma_1, \Psi, \Omega_{\varepsilon\varepsilon})' \in \Theta$ is a vector of structural parameters of the model as well as the covariance matrix of the exogenous shocks. The matrices Γ_0 and Γ_1 are of dimension $k \times k$, possibly singular, and the matrices Ψ and Π are of dimension $k \times \ell$ and $k \times p$, respectively. Also, we assume

$$E_{t-1}(\varepsilon_t) = 0, \quad \text{and} \quad E_{t-1}(\eta_t) = 0.$$

We also define the $\ell \times \ell$ matrix

$$\Omega_{\varepsilon\varepsilon} \equiv E_{t-1}(\varepsilon_t \varepsilon_t'),$$

which represents the covariance matrix of the exogenous shocks.

Consider a model whose maximum degree of indeterminacy is denoted by m.[5] The proposed methodology appends to the original LRE model in (8) the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix} \tag{9}$$

where the vector $\eta_{f,t}$ is a subset of the endogenous shocks and the vectors $\{\omega_t, \nu_t, \eta_{f,t}\}$ are of dimension $m \times 1$. The equations in (9) are autoregressive processes whose innovations are linear

[5] Denoting by n the minimum number of unstable roots of a LRE model and p the number of one-step ahead forecast errors, the maximum degrees of indeterminacy are defined as $m \equiv p - n$. When the minimum number of unstable roots of a model is unknown, then m coincides with number of expectational variables p. This represents the maximum degree of indeterminacy in any model with p expectational variables.

combinations of a vector of newly defined sunspot shocks, ν_t, and a subset of forecast errors, $\eta_{f,t}$, where $E_{t-1}(\nu_t) = E_{t-1}(\eta_{f,t}) = 0$. As we will show below, the choice of which expectational errors to include in (9) does not affect the solution.

The intuition behind the proposed methodology works as in the example considered in the previous section. Let $m^*(\theta)$ denote the actual degree of indeterminacy associated with the parameter vector θ. Under indeterminacy the Blanchard-Kahn condition for the original LRE model in (8) is not satisfied. Given that the system is characterized by $m^*(\theta)$ degrees of indeterminacy, it is necessary to introduce $m^*(\theta)$ explosive roots to solve the model using standard solution algorithms. In this case, $m^*(\theta)$ of the diagonal elements of the matrix Φ are assumed to be outside the unit circle (in absolute value), and the augmented representation is therefore determinate because the Blanchard-Kahn condition is now satisfied. On the other hand, under determinacy the (absolute value of the) diagonal elements of the matrix Φ are assumed to be all inside the unit circle, as the number of explosive roots of the original LRE model in (8) already equals the number of expectational variables in the model ($m^*(\theta) = 0$). Also, in this case the augmented representation is determinate due to the stability of the appended auxiliary processes. Importantly, as shown for the univariate example in Section 2, the block structure of the proposed methodology guarantees that the autoregressive process, ω_t, never affects the solution for the endogenous variables, X_t.

Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the system in (8) and (9) can be written as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{10}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1(\theta) & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n(\theta) & \Pi_f(\theta) \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π in (8) is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p-m)$ and $k \times m$.[6]

We now show that the augmented representation of the LRE model delivers solutions which

[6] Suppose that $\underset{k\times 3}{\Pi} \equiv \begin{bmatrix} \underset{k\times 1}{\Pi_1} & \underset{k\times 1}{\Pi_2} & \underset{k\times 1}{\Pi_3} \end{bmatrix}$. The proposed augmented representation would therefore allow for the following three possible alternatives, $\hat{\Pi}_1 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ 0 & 0 & -1 \end{bmatrix}$, $\hat{\Pi}_2 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ 0 & -1 & 0 \end{bmatrix}$ and $\hat{\Pi}_3 \equiv \begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 \\ -1 & 0 & 0 \end{bmatrix}$. In the online Appendix, we show with an analytic example that the alternative representations have a unique mapping that ensures the equivalence among each of them.

in the determinate region of the parameter space, Θ^D, are equivalent to those obtained using standard solution algorithms, and in the indeterminate region, Θ^I, to those obtained using the methodology provided by Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015).[7]

3.1 Determinate equilibrium and equivalent characterizations

The characterization of a determinate equilibrium of the original system in (8) is a vector $\theta^D \in \Theta^D$. The characterization of the solution under determinacy using the proposed augmented representation is parametrized by the set of parameters θ^{BN} that combines the vector $\theta^D \in \Theta^D$ with the set of additional parameters $\theta_1 \in \Theta_1$, where the vector $\theta_1 \equiv vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})'$ collects the elements of the covariance matrix of the sunspot shocks, $\Omega_{\nu\nu}$, and the parameters of the covariances, $\Omega_{\nu\varepsilon}$, between the sunspot shock ν_t and the exogenous shocks ε_t.

Theorem 1 *Let θ^D and θ^{BN} be two parametrizations of a determinate equilibrium of the model*

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t.$$

For the BN equilibrium parametrized by θ^{BN}, the solution for the endogenous variables, X_t, is equivalent to the solution parameterized by θ^D and is independent of the additional parameters θ_1.

Proof. See Appendix A. ■

The intuition for this theorem can be understood by considering the determinate solution to the univariate example reported in Table 1. First, the endogenous variable, π_t, is only a function of the exogenous shock r_t, and *not* of the newly defined sunspot shock, ν_t. Similarly, the endogenous variables, X_t, of the original LRE model in (8) do not respond to sunspot shocks either as expected under determinacy. Second, the univariate example shows that under determinacy the appended system of equations constitutes a separate block which does not affect the dynamics of the endogenous variable, π_t. Similarly, the solution for the endogenous variables, X_t, constitutes a separate block relative to the auxiliary variables, ω_t, and is therefore independent of the additional parameters θ_1. Finally, as the latent processes do not affect the endogenous variables, X_t, the likelihood associated with a vector of observables that represents a linear transformation of the variables in X_t is invariant with respect to the method used to compute the solution.

[7] In order to simplify the exposition, when analyzing the case of indeterminacy we assume, without loss of generality, $m^*(\theta) = m$. As it will become clear, the case of $m^*(\theta) < m$ is a special case of what we present below.

3.2 Indeterminate equilibria and equivalent characterizations

The indeterminate equilibria found using the methodology of Lubik and Schorfheide (2003) are parametrized by two sets of parameters. The first set is defined by $\theta^I \in \Theta^I$. In addition, given that the system is indeterminate, Lubik and Schorfheide (2003) append additional m equations,

$$\underset{m\times\ell}{\widetilde{M}}\ \underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\ \underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2'}\ \underset{p\times 1}{\eta_t}\,. \tag{11}$$

Given the normalization $M_\zeta = I$ chosen by Lubik and Schorfheide (2004), the second set corresponds to $\theta_2 \in \Theta_2$, where $\theta_2 \equiv vec\left(\Omega_{\zeta\zeta}, \widetilde{M}\right)'$. Equation (11) introduces $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, $\Omega_{\zeta\zeta}$, and additional $m \times \ell$ parameters of the matrix $\widetilde{M}$ that is related to the covariances between η_t and ε_t.

The characterization of a Lubik-Schorfheide equilibrium is a vector $\theta^{LS} \in \Theta^{LS}$, where Θ^{LS} is defined as

$$\Theta^{LS} \equiv \left\{\Theta^I, \Theta_2\right\}.$$

Similarly, the full characterization of the solutions under indeterminacy using the proposed augmented representation is parametrized by the set of parameters $\theta^I \in \Theta^I$ common between the two methodologies, and the set of additional parameters $\theta_1 \in \Theta_1$. Using our approach, we also introduce $m \times (m+1)/2$ parameters associated with the covariance matrix of the sunspot shocks, $\Omega_{\nu\nu}$, and $m \times \ell$ parameters of the covariances, $\Omega_{\nu\varepsilon}$, between the sunspot shocks ν_t and the exogenous shocks ε_t.[8] A Bianchi-Nicolò equilibrium is characterized by a parameter vector $\theta^{BN} \in \Theta^{BN}$, where Θ^{BN} is defined as

$$\Theta^{BN} \equiv \left\{\Theta^I, \Theta_1\right\}.$$

The following theorem establishes the equivalence between the characterizations of indeterminate equilibria obtained by using the methodology in Lubik and Schorfheide (2003) and the proposed augmented representation.

Theorem 2 *Let θ^{LS} and θ^{BN} be two alternative parametrizations of an indeterminate equilibrium of the model*

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi\varepsilon_t + \Pi\eta_t.$$

For every BN equilibrium, parametrized by θ^{BN}, there exists a unique matrix $\widetilde{M}$ and a unique matrix $\Omega_{\zeta\zeta}$ such that $\theta_2 = vec(\Omega_{\zeta\zeta}, \widetilde{M})'$, and $\{\theta_1, \theta_2\} \in \Theta^{LS}$ defines an equivalent LS equilibrium.

[8] In Appendix A, we show how the normalization chosen by Lubik and Schorfheide (2004) maps one-to-one into a specific covariance matrix for the exogenous shocks under the methodology proposed in this paper.

Conversely, for every LS equilibrium, parametrized by θ^{LS}, there is a unique matrix $\Omega_{\nu\nu}$ and a unique covariance matrix $\Omega_{\nu\varepsilon}$ such that $\theta_3 = vec(\Omega_{\nu\nu}, \Omega_{\nu\varepsilon})'$, and $\{\theta_1, \theta_3\} \in \Theta^{BN}$ defines an equivalent BN equilibrium.

Proof. See Appendix A. ■

In the paper Farmer et al. (2015), the authors also show that their characterization of indeterminate equilibria is equivalent to Lubik and Schorfheide (2003). Therefore, the following corollary holds.

Corollary 1 *Given a parametrization θ^{BN} of a BN indeterminate equilibrium, there exists a unique mapping into the parametrization of an indeterminate equilibrium for Farmer et al. (2015), and vice-versa.*

Moreover, the following two considerations support Corollary 2 below, which describes a relevant result on the likelihood function of the augmented representation. First, as emphasized in this section, the solution of the model in the augmented state space has a block structure which ensures that the evolution of the endogenous variables in X_t is not a function of the autoregressive processes, ω_t. Second, note that the appended autoregressive processes in ω_t only serve the purpose of providing the necessary explosive roots under indeterminacy and creating a mapping from the sunspot shocks to the expectational errors. These auxiliary processes are not mapped into the observable variables through the measurement equation. These two considerations imply that the parameters of the matrix Φ introduced with the augmented representation are not identified. The algorithm only requires them to be inside or outside the unit circle. Corollary 2 then follows.[9]

Corollary 2 *Conditional on the existence of a solution, the likelihood function associated with the newly defined vector of endogenous variables, $\hat{X}_t$, does not depend on the additional parameters included in the augmented representation, Φ, and is equivalent to the likelihood function associated with the endogenous variables, X_t.*

While Subsection 3.1 shows that the augmented representation of the LRE model delivers solutions which under determinacy are equivalent to those obtained using standard solution algorithms, Theorem 2 proves that the indeterminate solutions of our methodology are equivalent to those obtained using Lubik and Schorfheide (2003, 2004) and Farmer et al. (2015). This

[9]Notice that Corollary 2 holds when the *augmented* representation has a unique solution. This happens in two cases. First, values of the structural parameters θ which guarantee determinacy in the original LRE model should be combined with values for α_i in the matrix Φ whose absolute value lies within the unit circle. Second, values of the structural parameters θ for which the original model is indeterminate should be combined with (absolute) values of α_i outside the unit circle.

theoretical result is crucial for the application of our methodology to the New-Keynesian (NK) model with rational bubbles of Galí (forthcoming) in Section 5.

3.3 Baseline solution

Our augmented representation parametrizes the continuum of equilibria under indeterminacy by introducing the standard deviation of the sunspot shocks included in the auxiliary processes, $\Omega_{\nu\nu}$, and their correlations with the fundamental disturbances, $\Omega_{\nu\varepsilon}$. In this section, we propose a baseline solution that arises naturally in the context of our solution method.

Lubik and Schorfheide (2004) propose a baseline solution that minimizes the distance between the impulse response functions of the model under indeterminacy and determinacy evaluated at the boundary of the region of determinacy. Our theoretical results show the equivalence between our methods and therefore the possibility of mapping their baseline solution into our representation. However, it is not always immediate to construct the baseline solution proposed by Lubik and Schorfheide (2004), given that the boundaries of the determinacy region are often unknown. What is more, in the context of our approach it is more natural to think about the baseline solution as the one that restricts to zero the *contemporaneous* impact of the fundamental shocks on the variables whose sunspot shocks are considered as drivers of the economy. In what follows, we aim at explaining how to interpret this baseline solution.

We start by considering the univariate example in Section 2. Table 1 shows that the indeterminate solution of the augmented representation takes the following form

$$\pi_t = \phi_\pi \pi_{t-1} - r_{t-1} + \nu_t, \qquad \omega_t = 0, \qquad \eta_t = \nu_t.$$

The indeterminate solution is such that the expectational variable, π_t, is predetermined and its contemporaneous deviations are only due to its sunspot shock, ν_t. In other words, the fundamental shock r_t can affect π_t only if it affects the sunspot shock ν_t. Thus, it seems natural to choose as baseline solution the one associated with no correlation between the sunspot shock and the fundamental shock, r_t. Such identification strategy equivalently implies that the fundamental shock does not have a contemporaneous impact on the inflation rate.

The key insight of the univariate example is the following: The indeterminate solution of our method is such that the expectational variables whose forecast error are included in the explosive auxiliary process *always* become predetermined under indeterminacy and their contemporaneous deviations from their steady state are *only* a function of their respective sunspot shocks. Therefore, our method naturally suggests a baseline solution that sets the correlations of the sunspot shocks with fundamental disturbances to zero. Such identification assumption implies that fundamental shocks do not have a contemporaneous impact on those expectational variables. Importantly, this identification strategy is reminiscent of the zero restrictions often used in the SVAR literature. At the same time, the identifying assumption implies that sunspot shocks are not in any way related to fundamental disturbances, a natural starting point for an economic analysis.

In a multivariate model, the same intuition follows. Following the notation in previous sections, consider a model whose maximum degree of indeterminacy is denoted by m and number of one-step ahead forecast errors for the expectational variables defined by p. Under indeterminacy, it holds that $m \leq p$, and our methodology is such that a researcher has to choose which non-fundamental shocks, $\eta_{f,t}$, to include in the auxiliary processes, therefore redefining them as drivers of the economy. As shown for the univariate example, such choice equivalently implies that under the indeterminate solution, the expectational variables corresponding to the forecast errors, $\eta_{f,t}$, are *always* predetermined and their contemporaneous changes are only explained by the associated sunspot shocks, ν_t.[10] In our approach, the baseline solution is pinned down by the identification strategy that sets the correlations of the sunspot shocks, ν_t, and fundamental shocks, ε_t, to zero. Such identification equivalently implies that none of the fundamental shocks can have a contemporaneous impact on the expectational variables.

As a multivariate example, consider the standard three-equation NK model of Lubik and Schorfheide (2004) that is characterized by one degree of indeterminacy, $m = 1$, and two expectational variables (inflation and output) such that $p = 2$. Our approach suggests to append one auxiliary process and choose one of the two possible forecast errors to include in this process. Suppose the expectational error on inflation is chosen. This choice equivalently implies that under indeterminacy the inflation rate is predetermined and contemporaneous deviations from the steady state only take place via the inflation sunspot shock. The baseline solution of our method is pinned down by setting the correlation of the forecast error with the fundamental shocks to zero, therefore assuming that the inflation rate responds to all the fundamental shocks with a lag. It is also relevant to point out that, given this specification, the other expectational variable, output, responds contemporaneously to all the fundamental shocks as it constitutes the forward-looking block of the model. Moreover, if the researcher would instead choose to include the forecast error for output in the auxiliary process, the corresponding baseline solution would be such that output would be predetermined and respond only with a lag to the fundamental shocks, while inflation would respond contemporaneously. Therefore, given the baseline solution in our method, the choice of which forecast error to include in the auxiliary process is equivalent to a choice about which expectational variable responds to the fundamental shocks of a model with a lag.

[10] To guarantee boudedness, the indeterminate solution of our approach always imposes $\nu_t = \eta_{f,t}$ such that the linear combination of non-fundamental shocks, $\eta_{f,t}$, and sunspot shocks, ν_t, is zero at any time t.

While the choice of *which* forecast errors, $\eta_{f,t}$, to include in the auxiliary processes delivers different economic assumptions, the choice of the *ordering* of the forecast errors *within* the vector, $\eta_{f,t}$, is irrelevant to determine the impact of fundamental shocks on the expectational variables. As we describe in Section 5, the model of Galí (forthcoming) displays up to two degrees of indeterminacy, $m = 2$, and has three expectational variables (inflation, output and an asset bubble) such that $p = 3$. Let us define the forecast error for inflation and output as $\eta_{\pi,t}$ and $\eta_{y,t}$ respectively. Conditioning on the choice of those two forecast errors to include in the auxiliary processes, the baseline solution constructed with either of the two alternative orderings $\eta^1_{f,t} = \{\eta_{\pi,t}, \eta_{y,t}\}$ and $\eta^2_{f,t} = \{\eta_{y,t}, \eta_{\pi,t}\}$ delivers an equivalent identification strategy: Both the inflation rate and output do not respond contemporaneously to the fundamental disturbances of the model.

4 Inference

In this section, we describe an Hybrid Metropolis-Hastings algorithm that can be paired with our solution method to facilitate inference in a model that allows for different degrees of indeterminacy. As discussed by Lubik and Schorfheide (2004), a researcher that estimates a model with different degrees of indeterminacy often faces the challenging situation in which the posterior can present jumps along the boundaries of the determinacy and indeterminacy regions. Furthermore, these models often present local peaks in the posterior distribution with the result that a Metropolis-Hastings random walk algorithm might end up gravitating around one of these local peaks. To alleviate these problems, we propose an hybrid algorithm that combines the standard Metropolis-Hastings random walk algorithm with a MCMC algorithm in which the proposal distribution is based on a mixture of normals centered on the different posterior modes. The idea of using an hybrid algorithm to improve the efficiency of the standard Metropolis-Hastings random walk algorithm is extensively discussed in Herbst and Schorfheide (2015) and Giordani et al. (2010). The algorithm proposed here builds on one specific example discussed in An and Schorfheide (2007). In what follows, we describe the key steps.

1. Using different starting values, apply a numerical optimization procedure to search for modes $\widetilde{\theta}_{(j)}$, $j = 1, ..., J$ of the posterior density. When the model allows for different degrees of indeterminacy, the search can be conditioned on determinacy or indeterminacy. This guarantees that each of the regions has a, possibly local, posterior mode.

2. For each mode, compute the inverse of the Hessian, denoted by $\widetilde{\Sigma}_{(j)}$, $j = 1, ..., J$.

3. Let $q_j(\theta)$ be the density of a multivariate distribution obtained mixing two normals, both with mean $\widetilde{\theta}_{(j)}$, but different covariance matrices $c^s_j\widetilde{\Sigma}_{(j)}$ and $c^l_j\widetilde{\Sigma}_{(j)}$, with $c^s_j < c^l_j$. Let z^l be

the probability of drawing from the normal with large variance:

$$q_j(\theta) = z^l N\left(\widetilde{\theta}_{(j)}, c_j^l \widetilde{\Sigma}_{(j)}\right) + \left(1 - z^l\right) N\left(\widetilde{\theta}_{(j)}, c_j^s \widetilde{\Sigma}_{(j)}\right)$$

4. Let π_j, $j = 1, ..., J$ be a set of probabilities and define $q(\theta)$ as:

$$q(\theta) = \textstyle\sum_{j=1}^{J} \pi_j q_j(\theta).$$

5. Choose a starting value $\theta^{(0)}$ for instance by generating a draw from $q(\theta)$.

6. For $s = 1, ..., nsim$, follow these steps:

   (a) Make a draw ϑ from the following proposal distribution:

$$\begin{aligned}
\widetilde{q}(\vartheta|\theta^{(s-1)}) &= w^{RW} N\left(\theta^{(s-1)}, c^{RW} \widetilde{\Sigma}_{(j)}\right) + \left(1 - w^{RW}\right) q(\theta) \\
&= w^{RW} N\left(\theta^{(s-1)}, c^{RW} \widetilde{\Sigma}_{(j)}\right) \\
&\quad + \left(1 - w^{RW}\right) \left[z^l N\left(\widetilde{\theta}_{(j)}, c_j^l \widetilde{\Sigma}_{(j)}\right) + \left(1 - z^l\right) N\left(\widetilde{\theta}_{(j)}, c_j^s \widetilde{\Sigma}_{(j)}\right)\right]
\end{aligned}$$

   where w^{RW} is a number between 0 and 1 denoting the probability of using the standard random walk proposal distribution.

   (b) Accept the jump from $\theta^{(s-1)}$ to ϑ $\left(\theta^{(s)} = \vartheta\right)$ with probability $\min\left\{1, r_j\left(\theta^{(s-1)}, \vartheta|Y\right)\right\}$, otherwise reject the proposed draw and set $\theta^{(s)} = \theta^{(s-1)}$, where

$$r_j\left(\theta^{(s-1)}, \vartheta|Y\right) = \frac{\mathcal{L}\left(\vartheta|Y\right) p\left(\vartheta\right) / \widetilde{q}(\vartheta|\theta^{(s-1)})}{\mathcal{L}\left(\theta^{(s-1)}|Y\right) p\left(\theta^{(s-1)}\right) / \widetilde{q}(\theta^{(s-1)}|\vartheta)}$$

Note that point 6 collapses to a standard Metropolis-Hastings algorithm if $w^{RW} = 1$, while it becomes similar to the hybrid MCMC algorithm proposed by An and Schorfheide (2007) to deal with a multimodal posterior if $w^{RW} = 0$. The use of the mixture of normals facilitates the jump between areas of the parameter space that gravitate around different peaks of the posterior. The advantage of allowing for the standard random walk proposal distribution is to allow the algorithm to explore the parameter space around these peaks in an efficient way. In other words, the standard random walk algorithm has generally a higher acceptance rate and does not face the risk of getting stuck on a value for which the ratio between the posterior and the proposal distribution is particularly high.

In our main application, we show that the standard Metropolis-Hastings, when combined with our solution method, can be used to estimate the model over the entire parameter space but it faces the risk of getting stuck in a local peak of the posterior. Instead, the hybrid algorithm guarantees an almost immediate transition to the region of the parameter space contained the global peak of the posterior. In Subsection 6.1, we show that if the difference between local and global peaks is not too large, the algorithm facilitates repeated jumps between the two regions, resulting in a much faster convergence of the MCMC chain.

5 Monetary Policy and Asset Bubbles

In this section, we implement the proposed methodology to estimate the small-scale NK model of Galí (forthcoming) using Bayesian techniques. The model extends a conventional NK model to allow for the existence of rational expectations equilibria with asset price bubbles. Interestingly, the model displays up to two degrees of indeterminacy for realistic parameter values.

We estimate the model using U.S. data over the period 1982:Q4 until 2007:Q3, and we consider the case that the U.S. monetary policy aimed at stabilizing the inflation rate and leaning against the bubble. We find that the strength of such responses was not enough to guarantee a stabilization of the U.S. economy and to avoid that unexpected changes in expectations could drive U.S. business cycles. In particular, we show that the model specification that provides the best fit to the data is characterized by two degrees of indeterminacy.[11]

5.1 The Model

The model of Galí (forthcoming) is described by the following equations. First, equation (12) represents a dynamic IS curve

$$y_t = \Phi E_t\left(y_{t+1}\right) - \Psi\left[i_t - E_t\left(\pi_{t+1}\right)\right] + \Theta q_t, \tag{12}$$

where the variables are expressed in deviations from a balanced growth path (henceforth BGP), and the parameters $\{\Phi, \Psi, \Theta\}$ are function of the structural parameters of the model.[12] The term q_t denotes the size of an aggregate bubble in the economy (normalized by trend output) relative to its value along the BGP.

[11] In line with our theoretical results, we show that, given the degree of indeterminacy, the estimation delivers the same results regardless of which forecast error we include in our representation since there exists a unique mapping among the alternative representations.

[12] In particular, $\Phi \equiv \frac{\Lambda\Gamma v}{\beta} \in (0,1]$, $\Psi \equiv \Upsilon\Phi\left(1 + \frac{v\gamma(1-\Phi)}{\Phi(1-\beta\gamma)}\right)$, $\Upsilon \equiv \frac{1-\beta\gamma}{1-\Lambda\Gamma v\gamma} \in (0,1]$ and $\Theta \equiv \frac{(1-\beta\gamma)(1-v\gamma)}{\beta\gamma}$. The parameters are function of the following structural parameters of the model: i) γ, the constant probability of each individual in the OLG model to survive to the next period; ii) v, the probability of each individual to be employed in the next period; iii) β, the discount factor of each individual; iv) $\Lambda \equiv 1/(1+r)$, the steady state stochastic discount factor for one-period ahead payoffs derived from a portfolio of securites; v) $\Gamma \equiv (1+g)$, the gross rate of productivity growth.

The aggregate bubble plays the role of demand shifter and is defined as

$$q_t = b_t + u_t^q, \tag{13}$$

where b_t denotes the aggregate value in period t of bubble assets that were already available for trade in period $t-1$, and u_t^q is the value of a new bubble at time t. We assume that u_t^q follows an exogenous autoregressive process of the form

$$u_t^q = \rho_q u_{t-1}^q + \varepsilon_t^q, \qquad \varepsilon_t^q \overset{iid}{\sim} N(0, \sigma_q^2).$$

Equation (14) defines the evolution of the value of the asset bubble q_t as

$$q_t = \Lambda\Gamma E_t\left(b_{t+1}\right) - q\left(i_t - E_t\left(\pi_{t+1}\right)\right), \tag{14}$$

where $q \equiv \frac{\gamma(\beta-\Lambda\Gamma v)}{(1-\beta\gamma)(1-\Lambda\Gamma v\gamma)}$ represents the steady state bubble-to-output ratio, $\Lambda \equiv 1/(1+r)$ is the steady state stochastic discount factor for one-period ahead payoffs derived from a portfolio of securities and $\Gamma \equiv (1+g)$ is the gross rate of productivity growth. Equation (14) shows how "optimistic" expectations about the future value of the bubble lead to a higher price for the assets today. As shown in Galí (forthcoming), the existence of a BGP with positive asset bubbles requires that $\Lambda\Gamma v < \beta$. In addition, to guarantee that newly created bubbles along the BGP are non-negative, the model requires that $\Lambda\Gamma \geq 1$. Equivalently, these two conditions imply that there exists a continuum of bubbly BGPs indexed by a real interest rate in the range $\Gamma v/\beta - 1 < r \leq g$.

The model features a NK Phillips curve

$$\pi_t = \Lambda\Gamma v\gamma E_t\left(\pi_{t+1}\right) + \kappa y_t + u_t^s, \tag{15}$$

where $u_t^s = \rho_s u_{t-1}^s + \varepsilon_t^s$ and $\varepsilon_t^s \overset{iid}{\sim} N(0, \sigma_s^2)$.[13] Finally, the conduct of monetary policy follows an interest rate rule according to which monetary policy aims not only at stabilizing inflation, but also at leaning against the bubble:

$$i_t = \phi_\pi \pi_t + \phi_q q_t + \varepsilon_t^i, \tag{16}$$

where $\varepsilon_t^i \overset{iid}{\sim} N(0, \sigma_i^2)$.

Equations (12)∼(16) describe the equilibrium dynamics of the model economy around a given

[13] In particular, $k \equiv \frac{(1-\theta)(1-\Lambda\Gamma v\gamma\theta)}{\theta}\rho$, where θ represents the Calvo probability that a firm keeps its price unchanged in any given period and ρ is the elasticity of hours worked.

BGP. We define the vector of variables as $X_t \equiv (y_t, \pi_t, b_t, i_t, q_t, E_t(y_{t+1}), E_t(\pi_{t+1}), E_t(b_{t+1}), u_t^q, u_t^s)'$, and the vectors of fundamental shocks, ε_t, and non-fundamental errors, η_t, as

$$\varepsilon_t \equiv \left(\varepsilon_t^q, \varepsilon_t^s, \varepsilon_t^i\right)', \qquad \eta_t \equiv \left(\eta_{y,t}, \eta_{\pi,t}, \eta_{b,t}\right)'.$$

where the rational expectation forecast errors are defined as

$$\eta_{y,t} \equiv y_t - E_{t-1}[y_t], \qquad \eta_{\pi,t} \equiv \pi_t - E_{t-1}[\pi_t], \qquad \eta_{b,t} \equiv b_t - E_{t-1}[b_t]. \tag{17}$$

The model can therefore be represented as

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{18}$$

where θ represents the vector of structural parameters of the model.

Galí (forthcoming) shows that for realistic parameter values, the model is characterized by up to two degrees of indeterminacy. Therefore, the proposed methodology augments the representation of the model in (18) with two autoregressive processes

$$\omega_{1,t} = \left(\frac{1}{\alpha_1}\right)\omega_{1,t-1} + \nu_{1,t} - \eta_{1,t}, \tag{19}$$

$$\omega_{2,t} = \left(\frac{1}{\alpha_2}\right)\omega_{2,t-1} + \nu_{2,t} - \eta_{2,t}, \tag{20}$$

where $\{\eta_{1,t}, \eta_{2,t}\}$ could be any combination consisting of two of the three forecast errors defined by the vector $\eta_t \equiv \left(\eta_{y,t}, \eta_{\pi,t}, \eta_{b,t}\right)'$. Hence, we define a new vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_{1,t}, \omega_{2,t})'$ and a newly defined vector of exogenous shocks as $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_{1,t}, \nu_{2,t})'$. The system in (18), (19) and (20) can then be written as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t.$$

5.2 Data and priors

We estimate the model to match U.S. data over the period 1982:Q4 until 2007:Q3. We consider three macroeconomic quarterly time series: the growth rate in real GDP, measured as the log change in real GDP, inflation, measured by the log change in the GDP deflator, and the Federal Funds rate. The measurement equations that relate the macroeconomic data to the endogenous variables are defined as

$$\begin{bmatrix} \Delta \log (GDP_t) \\ \Delta \log (P_t) \\ FFR_t \end{bmatrix} = \begin{bmatrix} g \\ \pi^* \\ r + \pi^* \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ \pi_t \\ i_t \end{bmatrix},$$

Table 2 reports the prior distributions for the parameters. We calibrate three parameters to guarantee identification. Following Galí (forthcoming), we calibrate the discount factor of each individual, β, to 0.998 and the probability of surviving to the next period, γ, to 0.996. As mentioned when studying equation (14) describing the evolution of the value of the asset bubble q_t, the model requires that the real interest rate, r, and the growth rate of output, g, satisfy $r \leq g$ to ensure that newly created bubbles along the BGP are non-negative. To ensure that this inequality holds for each draw of the Metropolis-Hastings algorithm, we express the real interest rate, r, as $r = \lambda_u g$, where $\lambda_u \in (0, 1]$ defines the ratio between the real interest rate and its upper bound, the rate of output growth. We then calibrate λ_u to 0.925 and set the prior for the quarterly growth rate of output, g, as a gamma distribution centered at 0.45. These assumptions imply an annualized growth rate of output of 1.8% and real interest rate of approximately 1.65% over the considered period.

As previously discussed, the existence of a BGP with positive asset bubbles requires that $v < \beta/\Lambda\Gamma$, where v is the probability that an individual remains "active" by supplying labor and managing the firm, as opposed to "retiring" with probability $(1 - v)$. Hence, we express such probability as $v = \lambda_l \beta/\Lambda\Gamma$, where $\lambda_l \in (0, 1)$. We then center the gamma prior distribution for the term $100(\lambda_l^{-1} - 1)$ such that the probability of remaining "active," v, is 0.9973, therefore coinciding with the calibration in Galí (forthcoming). The resulting range of admissible BGPs is indexed by an (annualized) real interest rate $r \in (\Gamma v/\beta - 1, g] = (1.5\%, 1.8\%]$.

The prior for the slope of the New Keynesian Phillips Curve, κ, is set at 0.04, a value chosen for the calibration in Galí (forthcoming) and consistent with an average duration of individual prices of 4 quarters in this model. The parameter describing the response of the monetary authority to changes in inflation, ϕ_π, follows a gamma distribution with mean 1 and standard error 0.4. The response to deviations of the bubble relative to its value along the BGP, ϕ_q, follows a gamma distribution with mean 0.05 and standard error 0.02, corresponding to a region of the parameter space which allows for various degrees of indeterminacy, as shown in Galí (forthcoming).

The prior distribution of the supply and monetary policy shocks are inverse gamma centered at 0.3 with a standard deviation of 0.15. The inverse gamma prior for the shock associated with the creation of a new bubble, ε_t^q, is more agnostic and centered at 1 with standard deviation 0.5. Finally, when we estimate the model under indeterminacy, we specify uniform prior distributions

Table 2: Prior and posterior distributions of model parameters

	Posteriors		Priors		
	Mean	90% prob. int.	Density	Mean	Std. Dev.
$100(\lambda_l^{-1}-1)$	0.026	[0.016,0.036]	*Gamma*	0.04	0.01
κ	0.038	[0.030,0.046]	*Gamma*	0.04	0.005
g	0.47	[0.42,0.53]	*Gamma*	0.45	0.04
π^*	0.91	[0.48,1.46]	*Gamma*	0.9	0.30
ϕ_π	0.32	[0.16,0.54]	*Gamma*	1	0.40
ϕ_q	0.04	[0.02,0.08]	*Gamma*	0.05	0.02
σ_q	1.51	[0.75,2.60]	*Inv. Gam.*	1.00	0.50
σ_s	0.12	[0.10,0.15]	*Inv. Gam.*	0.30	0.15
σ_i	0.13	[0.10,0.16]	*Inv. Gam.*	0.30	0.15
ρ_q	0.80	[0.67,0.91]	*Beta*	0.70	0.10
ρ_s	0.88	[0.79,0.94]	*Beta*	0.70	0.10
σ_{ν_π}	0.28	[0.23,0.32]	$U[0,10]$	5	2.89
σ_{ν_y}	0.71	[0.63,0.81]	$U[0,10]$	5	2.89
$\varphi_{\nu_\pi,i}$	-0.59	[-0.70,-0.46]	$U[-1,1]$	0	0.57
$\varphi_{\nu_\pi,q}$	0.43	[0.18,0.68]	$U[-1,1]$	0	0.57
$\varphi_{\nu_\pi,s}$	0.62	[0.44,0.73]	$U[-1,1]$	0	0.57
$\varphi_{\nu_y,i}$	-0.37	[-0.57,-0.15]	$U[-1,1]$	0	0.57
$\varphi_{\nu_y,q}$	0.22	[-0.15,0.57]	$U[-1,1]$	0	0.57
$\varphi_{\nu_y,s}$	-0.63	[-0.73,-0.49]	$U[-1,1]$	0	0.57

The table reports the prior and posterior distributions under two degrees of indeterminacy $\{\nu_\pi, \nu_y\}$.

for the standard deviations of the non-fundamental shocks $\{\sigma_{\nu_l}\}$ where $l = \{\pi, y, b\}$, and their correlations with the exogenous shocks of the model $\{\varphi_{\nu_{l,j}}\}$ where $j = \{i, q, s\}$. The joint prior for the covariance matrix is effectively truncated to ensure that the covariance matrix is always positive definite (parameter draws that violate this condition are automatically rejected).

5.3 Results

We estimate the model using the hybrid algorithm described in Section 4. First, we use different starting values and apply a numerical optimization procedure to search for modes of the posterior density conditioning on each region of the parameter space: Determinacy, one degree of indeterminacy, and two degrees of indeterminacy. Each region has a, possibly local, posterior mode that we then use to construct the proposal distribution as in step 6 of the hybrid algorithm. In the following, we consider the specification in which the two auxiliary processes in (19) and (20) include the non-fundamental errors associated with inflation and output, $\{\eta_{1,t}, \eta_{2,t}\} = \{\eta_{\pi,t}, \eta_{y,t}\}$. When the algorithm draws a vector of structural parameters θ such that the model is indeterminate of degree 1, we set only α_1 to a value within the unit circle in a way that only the non-fundamental shock $\eta_{\pi,t}$ is redefined as fundamental. When the algorithm draws θ such that the model is indeterminate of degree 2, we set both α_1 and α_2 to a value within the unit circle such that both non-fundamental shocks $\{\eta_{\pi,t}, \eta_{y,t}\}$ are redefined as fundamental. In line with our theoretical results, we show in Appendix B that the estimation delivers the same posterior distributions of the model parameters regardless of which forecast errors we include in our representation.[14]

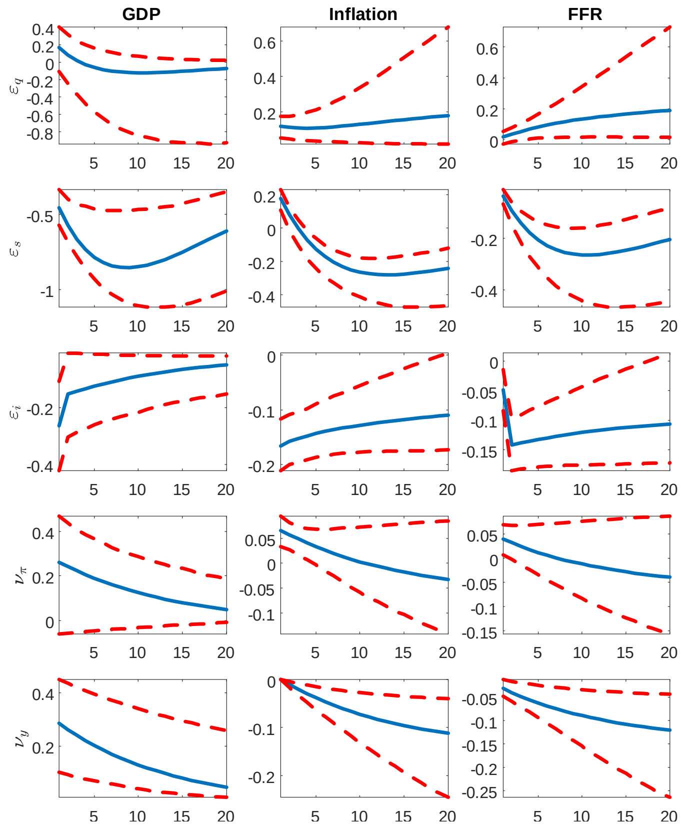
Using starting values in each region of the parameter space, we find that the data favor the specification of the model with two degrees of indeterminacy. In other words, the algorithm quickly moves to that region of the parameter space and never leaves. This can be explained inspecting the log-posterior mode of the different regions of the parameters space: -32.19 with two degrees of indeterminacy, -51.49 with one degree of indeterminacy, and -211.23 with determinacy. We attribute this result to the stylized nature of the model, and the observation that indeterminate models are consistent with a richer dynamic and stochastic structure. In future work, it would be valuable to study whether the findings would carry over in the context of a richer, medium-scale model that could explain the persistence and volatility in the data without recurring to indeterminate dynamics. In Subsection 6.1, we use the model of Lubik and Schorfheide (2004) to show that when the regions of the parameter space present a more similar fit, the hybrid algorithm facilitates the transition between them.

Table 2 also reports the mean and 90% probability interval of the posterior distribution of the estimated structural parameters. The posterior mean of the slope of the NK Phillips curve is 0.038, which in this model is consistent with a probability of roughly 25% that a firm keeps its price unchanged in any given period. The steady-state quarterly growth rate of output, g, is about 0.47% and the resulting real interest rate, r, is 0.43%. The posterior mean for the inflation rate, π^*, is about 3.6% on an annual basis. The strength of the responses of U.S. monetary policy to inflation and the bubble was not enough to guarantee a stabilization of the U.S. economy and to avoid that unexpected changes in expectations could drive U.S. business cycles. The posterior mean of the term λ_l is 0.9997 such that probability of remaining "active" is $v = \lambda_l \beta / \Lambda\Gamma = 0.997$.

The mean of the standard error of the bubble component is 1.51, and larger than the standard deviation of the supply and monetary policy shocks that are estimated to be 0.11 and 0.12, respectively. The data also provide evidence that the bubble shock is slightly less persistent than the supply shock. We also report the standard deviation of the sunspot shocks for the representation that includes the forecast error for the output gap and the inflation rate. The posterior estimates show that the standard error related to forecast errors for the output gap is

[14] The posterior distributions of the model parameters are equivalent up to a transformation of the correlations between the exogenous shocks and the sunspot disturbances considered in each specification.

Figure 1: Bayesian impulse response functions.



Each panel plots the posterior mean (solid lines) and 90-percent probability interval (dashed lines) for the impulse responses of output, inflation and Federal Funds rate (FFR) to a shock of one standard deviation for each orthogonalized disturbance using a Cholesky decomposition with the same order as in the figure.

about twice as large as the standard deviation of the sunspot shock associated with the inflation rate.

Finally, the data are informative on the correlations of both sunspot shocks with the exogenous shocks of the model. A monetary policy shock is negatively correlated with both sunspot shocks, implying a contemporaneous impact on both inflation and output. A shock due to a new bubble can be interpreted as a demand shifter, and it has a positive significant correlation only with unexpected changes in expectations about future inflation. A supply shock has a positive correlation with the sunspot shock associated with inflation, as well as a negative correlation with the sunspot shock for output. These correlations are crucial to interpret the impact that each shock has on the model economy, as described next.

Table 3: Summary statistics for the distribution of number of draws necessary to cross to the indeterminacy-2 region.

Starting value	Algorithm	Mean	Median	90% Prob.Int.	Pr(jump>10^5)
Determinacy	Mixture	18.0	8	[1, 68]	0
	Random walk	36.0	12	[1, 143.5]	0
Indeterminacy-1	Mixture	30.7	22	[2, 90.5]	0
	Random walk	3406.5	811.5	[36, 6911]	1.8%

The table reports summary statistics for the distribution of the number of draws necessary to cross to the region with two degrees of indeterminacy for different starting values and using both the hybrid algorithm ("Mixture") and the standard Metropolis-Hastings random walk algorithm ("Random walk").

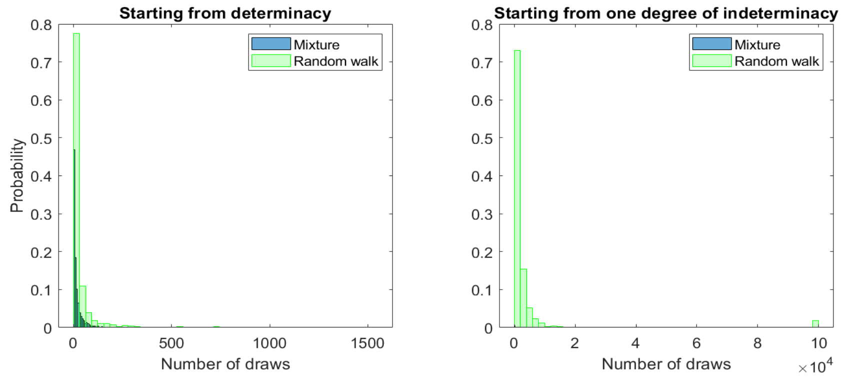
Figure 1 plots the impulse response functions of output, inflation, and the FFR to a one-standard-deviation shock of the fundamental and sunspot shocks. The solid lines represent the posterior means, while the dashed line correspond to the 90% probability intervals. In line with the estimated correlations reported in Table 2, we observe that a shock due to the creation of a new bubble generates a positive contemporaneous and persistent effect on inflation. A positive supply shock has both inflationary and contractionary effects on impact. The persistence of the shock on output is then associated to deflationary effects to which the monetary authority responds by decreasing the nominal interest rate. A monetary policy tightening generates contractionary and deflationary pressures and lower current and expected values of the asset bubble. The persistence of these effects on the real economy and the inflation rate requires the monetary authority to adopt an accommodative stance.

The last two panels show the impulse responses to the sunspot shocks, that we assume to be uncorrelated with each other. In this economy, a positive shock to inflation expectations, ν_π, generates self-fulfilling inflationary effects, leading to a rise in economic activity and a stabilizing monetary policy tightening. Finally, a positive sunspot shock to the expectation about future output leads to a rise in economic activity due to its self-fulfilling nature. Given the sunspot shock, ν_y, is assumed to be uncorrelated with the sunspot shock, ν_π, the inflation rate does not respond on impact to a one-standard deviation shock to ν_y, while it is characterized by a mild deflationary effect in the medium term that leads to a decrease in the nominal interest rate.

6 Practical implementation

In this section, we discuss the advantages of using the hybrid estimation algorithm and propose some tips about using our solution method with more standard software packages.

Figure 2: Distribution of number of draws necessary to cross to the indeterminacy-2 region.



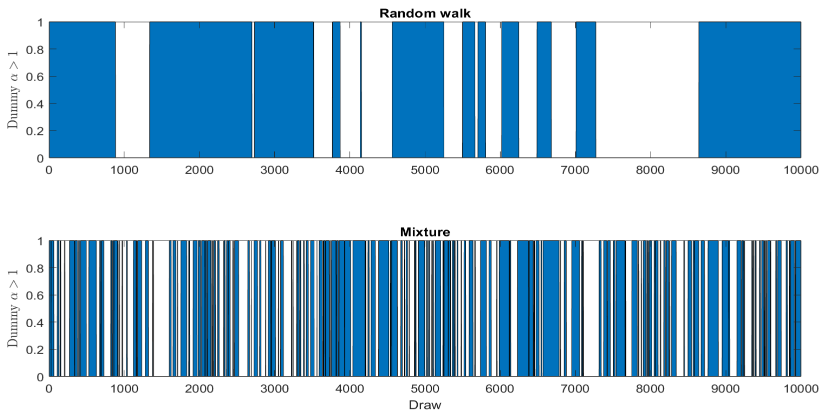
The figure reports the distribution for the number of draws necessary to cross the determinacy and indeterminacy of degree 1 threshold for the first time when using two distinct algorithms to estimate the model of Galí (2018). Two cases are considered. In the first case (blue/dark colored bars), we use the hybrid algorithm ("Mixture") described in Section 4. In the second case (green/light colored bars), we use a standard Metropolis-Hastings random walk algorithm ("Random walk").

6.1 Advantages of the hybrid algorithm

The estimation of a model with various degrees of indeterminacy could be a challenging task. The hybrid algorithm ("Mixture") proposed in Section 4 provides valuable advantages that alleviate these challenges relative to a standard Metropolis-Hasting random walk algorithm ("Random walk"). When estimating the model, the posterior could be characterized by discontinuities at the boundary of the (in)determinate regions that could prevent a smooth transition between areas of the parameter space that have similar fit.

A researcher could then face two cases. First, the fit of the model at the global peak that is found in a given region of the parameter space substantially outperforms the fit at a local peak associated with a different region of determinacy. In this case, the hybrid algorithm speeds up the transition to the correct region of the parameter space and prevents the MCMC algorithm from getting stuck in a region of the parameter space that only contains a local peak of the posterior. The estimation of the model in Galí (forthcoming) is an example of such case as we find peaks of the posterior in each of the three different regions of the parameter space. To understand how the Mixture algorithm helps to ensure a more efficient convergence to the global peak, we estimate the model using both algorithms. We simulate 2,000 chains by making draws from the posterior mode in either the region of determinacy or indeterminacy of degree 1. For each iteration, we count the number of draws necessary for the parameters to cross the two-degree indeterminacy threshold for the first time. If the transition has not occurred after 100,000 iterations, we stop

Figure 3: Jumps between determinacy and indeterminacy in Lubik and Schorfheide (2004).



The figure reports the draws of the parameters θ in the determinate and indeterminate region using the random walk (panel above) and hybrid (panel below) algorithms. A dummy variable has a value of 1 for each draw of parameters in the determinate region ($\alpha > 1$) and a value of 0 for draws in the indeterminate region ($\alpha \leq 1$). The blue/dark area plots values of the dummy equal to 1, while the white area plots values equal to 0.

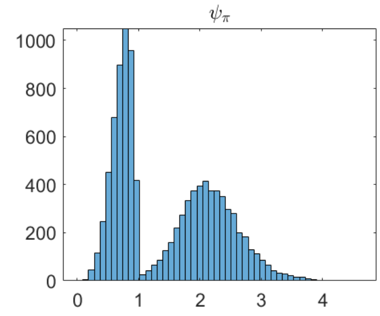
and record this upper bound. Using both the Mixture and the Random walk algorithms, Figure 2 reports the histogram of the number of draws necessary to jump to the region of two degrees of indeterminacy when starting from the region of determinacy (left panel) or indeterminacy of degree 1 (right panel). Table 3 reports the corresponding summary statistics of the plotted distributions.

Both algorithms eventually converge to the region of the parameter space that contains the global peak and do not jump back to explore the other regions where the fit of the model at the local peaks is substantially worse. However, the Mixture algorithm ensures a considerably faster switch to this region.[15] For starting values in the region of one-degree indeterminacy, the median number of draws necessary for the Mixture algorithm is more than 30 times smaller than the corresponding statistic for the Random walk algorithm. Instead, for the traditional algorithm, the parameters have *not* crossed the two-degree indeterminacy threshold after $100,000$ iterations in 1.8% of the cases.

The second case that a researcher could face is when the fit of the model at the global peak is only marginally better than that at the local peak in a different region of indeterminacy. In such instance, the advantage of the Mixture algorithm is to visit the different peaks of the

[15]In Appendix B, Table 5 reports the Raftery-Lewis diagnostics for each parameter chain in Galí (forthcoming). Using the hybrid algorithm, all the model parameters quickly converge.

Figure 4: Posterior distribution of ψ_π in Lubik and Schorfheide (2004).



The figure reports the posterior distribution of the parameter ψ_π when using the hybrid algorithm.

posterior in the different regions with a substantially higher frequency relative to the Random walk algorithm, allowing for a faster convergence of the MCMC algorithm. To provide an example of this second case, we estimate the three-equation NK model in Lubik and Schorfheide (2004) over the post-1982 period using our solution method and hybrid algorithm.[16] The parameter space is characterized by a region of determinacy and a region of indeterminacy of degree 1. To solve the model under indeterminacy, we augment it by appending only one auxiliary process. The global peak in the determinate region marginally outperforms the local peak in the region of one-degree indeterminacy. When estimating the model, we assign a value of 1 to a dummy variable for each draw of structural parameters in the determinate region ($\alpha > 1$) and a value of 0 for draws in the indeterminate region ($\alpha \leq 1$). Figure 3 reports values of the dummy equal to 1 (blue/dark areas) and evidently shows that, while both algorithms visit the two regions, the Mixture algorithm jumps between them much more frequently ensuring an efficient exploration of the parameter space. In Appendix B, Table 6 reports the Raftery-Lewis diagnostics for each parameter using the two algorithms: The Mixture algorithm cuts the number of draws required for convergency in half.[17]

The second case that we just considered is also frequently associated with multimodal posterior distributions of the model parameters. When estimating the model in Lubik and Schorfheide (2004) using the Mixture algorithm, Figure 4 plots the resulting bimodal distribution of the parameter ψ_π that governs the response of the monetary authority to deviations of the inflation rate from its target. In such cases, a researcher would need to cautiously verify the converge of the model parameters, a topic that we discuss in the next section.

[16] We refer the reader to Lubik and Schorfheide (2004) for a detailed description of the standard model.

[17] We target the 5% quantile, with 1% precision, and 90% probability.

6.2 Standard software packages

In this section, we consider the case that a researcher estimates a LRE model using Bayesian techniques and a conventional Metropolis-Hastings algorithm in Dynare. Let us consider the following LRE model

$$\Gamma_0(\theta)X_t = \Gamma_1(\theta)X_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t, \tag{21}$$

with a maximum degree of indeterminacy denoted by m.[18] As explained in detail in Section 3, the proposed methodology appends to the original LRE model the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \tag{22}$$

where Φ is a diagonal matrix whose entries are $\{1/\alpha_1, .., 1/\alpha_m\}$. Denoting the newly defined vector of endogenous variables $\hat{X}_t \equiv (X_t, \omega_t)'$ and the newly defined vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the resulting augmented LRE model can be written as

$$\hat{\Gamma}_0\hat{X}_t = \hat{\Gamma}_1\hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t. \tag{23}$$

Auxiliary autoregressive parameters. As a first step, we discuss how to handle the vector of additional autoregressive parameters, $\{\alpha_i\}_{i=1}^m$, introduced under our methodology. We can distinguish three cases:

1. When the threshold for the different regions of determinacy is known analytically, then α_i can be expressed as a function of the other parameters. In this case, there is no need to specify a prior on α_i and the prior probability of (in)determinacy is given by the prior on the parameter vector θ.

2. When the threshold is unknown and the researcher writes her own code, she can start with all the roots inside the unit circle for α at each draw of θ and then flip the appropriate number of elements in the vector α. This case coincides with the approach that we adopt in Section 5 to estimate the model of Galí (forthcoming) for which there is no need to specify a prior on α_i and the prior probability of indeterminacy depends on the prior on the parameter vector θ. In other words, in this case α_i is treated as an unknown transformation of the structural parameters that guarantees that a solution, if it exists, can be found independently of the degree of indeterminacy.

3. When the threshold is unknown and the researcher wants to use standard estimation packages such as Dynare, there are two options. First, the researcher estimates the model

[18] We refer the reader to Section 3 for definitions and notation.

separately for different degrees of indeterminacy. This is the simplest approach and we describe it more in detail below. Second, the researcher estimates the model over the whole parameter space. In this case, the researcher can complement the Dynare codes with a function that pins down the degrees of indeterminacy. This can be done writing a function that, starting with all $\{\alpha_i\}_{i=1}^m$ inside the unit circle, solves the model and keeps flipping $\{\alpha_i\}_{i=1}^m$ until the augmented state space returns determinacy. In this case, the $\{\alpha_i\}_{i=1}^m$ are still treated as a transformation of the structural parameters of the model. Alternatively, the researcher can decide to treat $\{\alpha_i\}_{i=1}^m$ as additional parameters. In this case, the researcher should choose priors that are symmetric with respect to the various determinacy regions and orthogonal with respect to the priors on the other parameters. The researcher could use a uniform distribution over the interval $[0.5, 1.5]$ or any symmetric interval around 1 as a prior distribution. The assumptions that the priors are symmetric around 1 and orthogonal with respect to the structural parameters imply that the a-priori probabilities of the different determinacy regions only depend on the priors on the structural parameters of the model. The posterior distribution of the parameters is not affected by treating $\{\alpha_i\}_{i=1}^m$ as additional parameters. However, the priors on $\{\alpha_i\}_{i=1}^m$ would have an impact on the Marginal Data Density computed by Dynare. The Marginal Data Density can be corrected ex-post by taking into account that uniform priors on $\{\alpha_i\}_{i=1}^m$ simply rescale the joint prior on the model parameters. Alternatively, a researcher could implement a simple modification of the code used to compute Geweke's Modified Harmonic Mean to remove the impact of the priors on α_i. For example, $\{\alpha_i\}_{i=1}^m$ could be weighted using their own prior when computing the Modified Harmonic Mean.

Priors for the correlations between the sunspot and fundamental shocks In Subsection 3.3, we discuss in detail the economic rationale for how to construct a baseline solution using our methodology. Therefore, it seems natural to center the prior distribution for the correlations on zero, the value associated with the "baseline solution." However, as carefully explained in Subsection 3.3, it is important to stress that under the baseline solution the choice of which forecast errors to include in the auxiliary processes matters for the solution when the correlations are restricted to zero. At the same time, our theoretical results show that a set of correlations under the representation that includes a given subset of non-fundamental shocks has a unique mapping to *different* values of the correlations in the representation with another subset of non-fundamental disturbances, and vice versa. Therefore, in order for the alternative representations to deliver the same fit to the data, a researcher has to leave the correlations unrestricted. One simple option is to set a uniform prior distributions over the interval $(-1, 1)$ for the correlations of the sunspot shocks. As shown for the estimation of the model of Galí (forthcoming) in Section 5, this approach guarantees that the fit of the model does not depend on which non-fundamental shock is included in the auxiliary processes. Of course, if a researcher has reasons to believe that one baseline solution is more meaningful than the other, she can choose the priors accordingly.

Lubik and Schorfheide (2004) center the prior distribution for the additional parameters introduced in their representation to values that minimize the distance between the impulse responses of the model under indeterminacy and determinacy evaluated at the boundary of the region of determinacy. While our approach and intuition differ, our theoretical results show the equivalence between the two representations in Section 3. Therefore, the priors for the correlations between sunspot shocks and fundamental shocks could also be specified in a way to replicate the approach of Lubik and Schorfheide (2004). Specifically, we could center the prior on the auxiliary parameters as in Lubik and Schorfheide (2004) and then map those values into correlations in our approach that would deliver the same fit of the model to the data. However, our suggestion to choose a flat prior such as a uniform distribution considers a-priori the mapped parameterization suggested by Lubik and Schorfheide (2004) as equally likely with respect to the continuum of indeterminate equilibria.

Only (in)determinacy. In some cases, a researcher might want to estimate the model exclusively under determinacy or exclusively under indeterminacy. Our approach easily accommodates this need. If the researcher is only interested in the solution under determinacy, the parameter vector of α_i should be chosen in a way to guarantee stationarity of the auxiliary process (for example, fixing the value of all the α_i to 2). Furthermore, all parameters that are relevant only under indeterminacy should be fixed to zero or any other constant, given that they do not affect the fit of the model under determinacy. If instead the researcher is only interested in estimating the model under indeterminacy, the parameters of the auxiliary process can be chosen in a way to guarantee that the correct number of explosive roots are provided. In this case, the parameters describing the properties of the sunspot disturbances should also be estimated.

Model comparison. A researcher might also be interested in comparing the fit of the model under determinacy and under indeterminacy. Model comparison can be conducted by using standard techniques, such as the harmonic mean estimator proposed by Geweke (1999). If the researcher is interested in comparing the same model under determinacy and under indeterminacy, we recommend the following procedure that adapts the approach used by Lubik and Schorfheide (2004):

1. Estimate the model under determinacy by fixing the parameter(s) α_i to a value larger than one in a way that the model is solved only under determinacy. Note that in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks and its correlations with the exogenous shocks, *should* be restricted to zero (or any other constant). This restriction avoids penalizing the model for extra parameters

that do not affect its fit under determinacy.

2. Estimate the model under indeterminacy by fixing the parameter(s) α_i to a value smaller than one in a way that the model is solved only under indeterminacy. Note that in this case all parameters that pertain to the solution under indeterminacy, such as the volatility of the sunspot shocks and its correlations, should be estimated.

3. Use standard methods to compare the fit of the model under determinacy with the fit of the same model under indeterminacy.

Convergence. In Subsection 6.1, we show that, even if not efficiently, a standard estimation algorithm such as the one implemented in Dynare could travel to the correct region of the parameter space. At the same time, we want to emphasize the importance of conducting standard convergence diagnostics. As an example, consider the second case described in Subsection 6.1 when we estimate the model in Lubik and Schorfheide (2004). Figure 4 shows that the posterior distribution of the parameter ψ_π is bimodal. This in general would imply an increase in the number of draws required for convergence.

A researcher should then appropriately verify the occurrence of either of the following two circumstances. This bimodal distribution could arise because the log-likelihood is highly discontinuous between the two regions. In this case, the algorithm could have jumped towards the region where the peak of the posterior lies, without having spent a significant time there. In other words, convergence has not occurred yet. We therefore recommend the researcher to analyze the draws of the parameter(s) α_i which have been accepted during the MCMC algorithm. By inspecting the behavior of the auxiliary parameter(s) α_i, a researcher can detect if the algorithm reached convergence or not. If the convergence has not occurred yet, the researcher should repeat the estimation exercise increasing the number of draws and making sure that the parameter(s) α_i stabilizes on one region of the parameter space.

In our example above, the posterior distribution plotted in Figure 4 could in principle be the result of the algorithm traveling across the two regions multiple times or evidence of lack of convergence. In the former case, the parameter(s) α_i would repeatedly transition between the two areas of the parameter space and could be used to infer the probability attached to determinacy. In our example, this hypothesis is confirmed by inspecting the draws of the α_i in Figure 3. In this case, the researcher could consider the adoption of more efficient algorithms, such as the hybrid algorithm described in Section 4 or in Herbst and Schorfheide (2015) to ensure a more efficient exploration of the entire parameter space.

7 Conclusions

In this paper, we propose a generalized approach to solve and estimate LRE models over the entire parameter space. Our approach accommodates both cases of determinacy and indeterminacy and it does not require the researcher to know the analytic conditions describing the region of determinacy or the degrees of indeterminacy.

When a LRE model is characterized by m degrees of indeterminacy, our approach augments it by appending m autoregressive processes whose innovations are linear combinations of a subset of endogenous shocks and a vector of newly defined sunspot shocks. We show that the solution for the resulting augmented representation embeds both the solution which is obtained under determinacy using standard solution methods and that delivered by solving the model under indeterminacy using the approach of Lubik and Schorfheide (2003) and equivalently Farmer et al. (2015).

We pair our solution method with an hybrid MCMC algorithm to estimate the small-scale NK model of Galí (forthcoming). Galí's model extends a conventional NK model to allow for the existence of rational bubbles. An interesting aspect of the model is that it displays up to two degrees of indeterminacy for realistic parameter values. We estimate the model using U.S. data over the period 1982:Q4 until 2007:Q3. We find that the data support the version of the model with two degrees of indeterminacy, implying that the central bank was not reacting strongly enough to the bubble component. The model of Galí (forthcoming) is quite stylized, but the results are intriguing and merit further exploration in future research. Finally, we show that our MCMC hybrid algorithm facilitates the transition to the correct area of the posterior and repeated jumps between local peaks of the posterior when these are close in value. These features minimize the changes to remain stuck in an area of the posterior characterized by a local peak and speed the convergence with respect to the standard random walk Metropolis-Hastings algorithm.

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8 Appendix

8.1 Appendix A

8.1.1 Equivalence under determinacy

This section considers the case in which the original LRE is determinate, and shows the equivalence of the solution obtained using the proposed augmented representation with the one from the standard solution method described in Sims (2001).

Canonical solution Consider the LRE model in (8) and reported in the following equation

$$\underset{k\times k}{\Gamma_0}\underset{k\times 1}{X_t} = \underset{k\times k}{\Gamma_1}\underset{k\times 1}{X_{t-1}} + \underset{k\times l}{\Psi}\underset{l\times 1}{\varepsilon_t} + \underset{k\times p}{\Pi}\underset{p\times 1}{\eta_t}\,. \tag{24}$$

The method described in Sims (2001) delivers a solution, if it exists, by following four steps. First, Sims (2001) shows how to write the model in the form

$$SZ'X_t = TZ'X_{t-1} + Q\Psi\varepsilon_t + Q\Pi\eta_t, \tag{25}$$

where $\Gamma_0 = Q'SZ'$ and $\Gamma_1 = Q'TZ'$ result from the QZ decomposition of $\{\Gamma_0, \Gamma_1\}$, and the $k \times k$ matrices Q and Z are orthonormal, upper triangular and possibly complex. Also, the diagonal elements of S and T contain the generalized eigenvalues of $\{\Gamma_0, \Gamma_1\}$.

Second, given that the QZ decomposition is not unique, Sims' algorithm chooses a decomposition that orders the equations so that the absolute values of the ratios of the generalized eigenvalues are placed in an increasing order, that is

$$|t_{jj}| \,/\, |s_{jj}| \geq |t_{ii}| \,/\, |s_{ii}| \quad \text{for } j > i.$$

The algorithm then partitions the matrices S, T, Q and Z as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad Z' = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where the first block corresponds to the system of equations for which $|t_{jj}| \,/\, |s_{jj}| \leq 1$ and the second block groups the equations which are characterized by explosive roots, $|t_{jj}| \,/\, |s_{jj}| > 1$.

The third step imposes conditions on the second, explosive block to guarantee the existence of at least one bounded solution. Defining the transformed variables

$$\xi_t \equiv Z' X_t = \begin{bmatrix} \underset{(k-n)\times 1}{\xi_{1,t}} \\ \underset{n\times 1}{\xi_{2,t}} \end{bmatrix},$$

where n is the number of explosive roots, and the transformed parameters

$$\widetilde{\Psi} \equiv Q'\Psi, \quad \text{and} \quad \widetilde{\Pi} \equiv Q'\Pi,$$

the second block can be written as

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1}(\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t).$$

As this system of equations contains the explosive roots of the original system, then a bounded solution, if it exists, will set

$$\underset{n\times 1}{\xi_{2,0}} = 0 \tag{26}$$

$$\underset{n\times \ell}{\widetilde{\Psi}_2} \underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\widetilde{\Pi}_2} \underset{p\times 1}{\eta_t} = 0, \tag{27}$$

where n also denotes the number of equations in (27). A necessary condition for the existence of a solution requires that the number of unstable roots (n) equals the number of expectational variables (p). In this section, we are considering the solution under determinacy, and this guarantees that there are no degrees of indeterminacy $m^*(\theta) = 0$. The sufficient condition then requires that the columns of the matrix $\widetilde{\Pi}_2$ are linearly independent so that there is at least one bounded solution. In that case, the matrix $\widetilde{\Pi}_2$ is a square, non-singular matrix and equation (27) imposes linear restrictions on the forecast errors, η_t, as a function of the fundamental shocks, ε_t,

$$\eta_t = -\widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \varepsilon_t. \tag{28}$$

The fourth and last step finds the solution for the endogenous variables, X_t, by combining the restrictions in (26) and (28) with the system of stable equations in the first block,

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1 \varepsilon_t + \widetilde{\Pi}_1 \eta_t) \\ &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1 \widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \right) \varepsilon_t \end{aligned} \tag{29}$$

Using the algorithm by Sims (2001), we can describe the solution under determinacy of the LRE model in (24) with equations (26), (28), and (29).

Augmented representation We now consider the methodology proposed in this paper, and we augment the LRE model in (24) with the following system of m equations

$$\omega_t = \Phi\omega_{t-1} + \nu_t - \eta_{f,t}, \qquad \Phi \equiv \begin{bmatrix} \frac{1}{\alpha_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \frac{1}{\alpha_m} \end{bmatrix}$$

where Φ is a $m \times m$ diagonal matrix. As the original model in (24) is determinate, then we assume that all the diagonal elements α_i belong to the interval $[1, 2]$. Therefore, we are appending a system of stable equations, and we show that the solution for the endogenous variables, X_t, is equivalent to the one found in Subsection 8.1.1. Defining the augmented vector of endogenous variables, $\hat{X}_t \equiv (X_t, \omega_t)'$ and the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$, the representation that we propose takes the form

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t, \tag{30}$$

where

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix},$$

and without loss of generality the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$, where the matrices Π_n and Π_f are respectively of dimension $k \times (p - m)$ and $k \times m$.

We can find a solution to the augmented representation in (30) by using Sims' algorithm. Similarly to the previous section, we follow the four steps which describe the algorithm. First, the solution algorithm performs the QZ decomposition of the matrices $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and the augmented representation takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{31}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$, and

$$\hat{S} = \begin{bmatrix} S_{11} & \mathbf{0} & S_{12} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & S_{22} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & \mathbf{0} & T_{12} \\ \mathbf{0} & \Phi & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T_{22} \end{bmatrix}, \qquad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Z_2 & \mathbf{0} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ Q_2 & \mathbf{0} \end{bmatrix}.$$

Importantly, note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z, Q\}$ found in the previous section using the canonical solution method.

Second, the algorithm chooses a QZ decomposition which groups the equations in a stable and an explosive block. Because this section assumes that the original model is determinate and that the diagonal elements of the matrix Φ are within the unit circle, the explosive block corresponds to the third system of equations in (31) which is characterized by explosive roots. Recalling the definition of the matrices $\hat{\Psi}$ and $\hat{\Pi}$, the system of equations in the third block is

$$\xi_{2,t} = S_{22}^{-1} T_{22} \xi_{2,t-1} + S_{22}^{-1} (\widetilde{\Psi}_2 \varepsilon_t + \widetilde{\Pi}_2 \eta_t). \tag{32}$$

The third step imposes conditions to guarantee the existence of a bounded solution. However, the explosive block in (32) is identical to the system of equations found in the previous section. Therefore, the algorithm imposes the same restrictions to guarantee the existence of a bounded solution, that is

$$\xi_{2,0} = 0 \tag{33}$$

and as found earlier

$$\eta_t = -\widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \varepsilon_t. \tag{34}$$

Finally, the last step combines these restrictions with the system of equations in the stable block which corresponds to the first and second systems of equations in (31),

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \left(\widetilde{\Psi}_1 - \widetilde{\Pi}_1 \widetilde{\Pi}_2^{-1} \widetilde{\Psi}_2 \right) \varepsilon_t, & (35) \\ \omega_t &= \Phi \omega_{t-1} + \nu_t - \eta_{f,t}. & (36) \end{aligned}$$

Recalling that $\xi_t \equiv Z' X_t$, the solution in (33)∼(36) obtained for the augmented representation of the LRE model delivers the same solution for the endogenous variables of interest, X_t, found in the previous section and defined in equations (26), (28), and (29).

8.1.2 Equivalence under indeterminacy

This section shows the equivalence of the solutions obtained for a LRE model under indeterminacy using the proposed augmented representation and the methodology of Lubik and Schorfheide (2003, 2004).

Lubik and Schorfheide (2003) As in Subsection 8.1.1, we consider the LRE model in (24) and reported below as (37)

$$\Gamma_0 X_t = \Gamma_1 X_{t-1} + \Psi \varepsilon_t + \Pi \eta_t. \tag{37}$$

In this section we assume that the model is indeterminate, and we present the method used by Lubik and Schorfheide (2003). The authors implement the first two steps of the algorithm by Sims (2001) and described in Subsection 8.1.1.[19] They proceed by first applying the QZ decomposition to the LRE model in (37) and then ordering the resulting system of equations in a stable and an explosive block as defined in equation (25). However, their approach differs in the third step when the algorithm imposes restrictions to guarantee the existence of a bounded solution. In particular, the restrictions in (26) and (27) reported below as (38) and (39) require that

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{38}$$

$$\underset{n\times\ell}{\widetilde{\Psi}_2}\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{\widetilde{\Pi}_2}\underset{p\times 1}{\eta_t} = 0. \tag{39}$$

Nevertheless, it is clear that the system of equation in (39) is indeterminate as the number of forecast errors exceeds the number of explosive roots ($p > n$). Equivalently, there are less equations (n) than the number of variables to solve for (p). To characterize the full set of solutions to equation (39), Lubik and Schorfheide (2003) decompose the matrix $\widetilde{\Pi}_2$ using the following singular value decomposition

$$\underset{n\times p}{\widetilde{\Pi}_2} \equiv \underset{n\times n}{U}\left[\begin{array}{cc}\underset{n\times n}{D_{11}} & \underset{n\times m}{\mathbf{0}}\end{array}\right]\underset{p\times p}{V'},$$

where m represents the degrees of indeterminacy. Given the partition $\underset{p\times p}{V} \equiv \left[\begin{array}{cc}\underset{p\times n}{V_1} & \underset{p\times m}{V_2}\end{array}\right]$, equation (39) can be written as

$$\underset{n\times n}{D_{11}^{-1}}\underset{n\times n}{U'}\underset{n\times\ell}{\widetilde{\Psi}_2}\underset{\ell\times 1}{\varepsilon_t} + \underset{n\times p}{V_1'}\underset{p\times 1}{\eta_t} = 0. \tag{40}$$

Given that the system is indeterminate, Lubik and Schorfheide (2003) append additional m equations,

$$\underset{m\times\ell}{\widetilde{M}}\underset{\ell\times 1}{\varepsilon_t} + \underset{m\times m}{M_\zeta}\underset{m\times 1}{\zeta_t} = \underset{m\times p}{V_2'}\underset{p\times 1}{\eta_t}. \tag{41}$$

The $m \times 1$ vector ζ_t is a set of sunspot shocks that is assumed to have mean zero, covariance matrix $\Omega_{\zeta\zeta}$ and to be uncorrelated with the fundamental shocks, ε_t, that is

$$E\left[\zeta_t\right] = 0, \quad E\left[\zeta_t\varepsilon_t'\right] = 0, \quad E\left[\zeta_t\zeta_t'\right] = \Omega_{\zeta\zeta}.$$

[19] It is relevant to mention that in this section the matrices obtained from the QZ decomposition and the ordering of the equations into a stable and an explosive block differ from those in Subsection 8.1.1 both in terms of their dimensionality and the elements constituing them. However, we opted to use the same notation for simplicity.

The matrix $\widetilde{M}$ captures the correlation of the forecast errors, η_t, with fundamentals, ε_t, and Lubik and Schorfheide (2003) choose the normalization $M_\zeta = I_m$. The reason to append the system of equations in (41) to the equations in (40) is to exploit the properties of the orthonormal matrix V. Premultiplying the system by the matrix V and recalling that $V * V' = I$, the expectational errors can be written as a function of the fundamental shocks, ε_t, and the sunspot shocks, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(-\underset{p\times n}{V_1} \underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1'} \underset{n\times \ell}{\widetilde{\Psi}_2} + \underset{p\times m}{V_2} \underset{m\times \ell}{\widetilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}.$$

More compactly,

$$\underset{p\times 1}{\eta_t} = \left(\underset{p\times n}{V_1} \underset{n\times \ell}{N} + \underset{p\times m}{V_2} \underset{m\times \ell}{\widetilde{M}} \right) \underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2} \underset{m\times 1}{\zeta_t}, \tag{42}$$

where

$$\underset{n\times \ell}{N} \equiv -\underset{n\times n}{D_{11}^{-1}} \underset{n\times n}{U_1'} \underset{n\times \ell}{\widetilde{\Psi}_2}.$$

is a function of the parameters of the model. Given the restriction in (38) and (42), the fourth step in the algorithm combines these equations with the system of stable equations in the first block as in Subsection 8.1.1,

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} (\widetilde{\Psi}_1 \varepsilon_t + \widetilde{\Pi}_1 \eta_t) \\ &= S_{11}^{-1} T_{11} \xi_{1,t-1} + S_{11}^{-1} \left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 V_1 N + \widetilde{\Pi}_1 V_2 \widetilde{M} \right) \varepsilon_t + S_{11}^{-1} \left(\widetilde{\Pi}_1 V_2 \right) \zeta_t. \end{aligned} \tag{43}$$

Using the method in Lubik and Schorfheide (2003), we can describe the solution for the original LRE model under indeterminacy with equations (38), (42) and (43).

Augmented representation We now consider the augmented representation as in (30) and reported below as

$$\hat{\Gamma}_0 \hat{X}_t = \hat{\Gamma}_1 \hat{X}_{t-1} + \hat{\Psi} \hat{\varepsilon}_t + \hat{\Pi} \eta_t, \tag{44}$$

where $\hat{X}_t \equiv (X_t, \omega_t)'$, $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$ and

$$\hat{\Gamma}_0 \equiv \begin{bmatrix} \Gamma_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Gamma}_1 \equiv \begin{bmatrix} \Gamma_1 & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{\Psi} \equiv \begin{bmatrix} \Psi & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{\Pi} \equiv \begin{bmatrix} \Pi_n & \Pi_f \\ 0 & -\mathbf{I} \end{bmatrix}.$$

where the matrix Π is partitioned as $\Pi = [\Pi_n \quad \Pi_f]$ without loss of generality.

The novelty of our approach is that, given our representation, we can easily obtain the solution by using Sims' algorithm even when the original LRE is assumed to be indeterminate. It is enough to assume that the auxiliary processes ω_t are characterized by explosive roots, or equivalently that the diagonal elements of the matrix Φ are outside the unit circle. This approach guarantees that the Blanchard-Kahn condition for the augmented representation is satisfied and, given the analytic form that we propose for the auxiliary processes, we show that the solution for the endogenous variables of interest, X_t, is equivalent to the method of Lubik and Schorfheide (2003).

To show this result, we simply apply the four steps of the algorithm described in Sims (2001) to the proposed augmented representation. First, the QZ decomposition of (44) takes the form

$$\hat{S}\hat{Z}'\hat{X}_t = \hat{T}\hat{Z}'\hat{X}_{t-1} + \hat{Q}\hat{\Psi}\hat{\varepsilon}_t + \hat{Q}\hat{\Pi}\eta_t, \tag{45}$$

where $\hat{\Gamma}_0 = \hat{Q}'\hat{S}\hat{Z}'$ and $\hat{\Gamma}_1 = \hat{Q}'\hat{T}\hat{Z}'$ result from the QZ decomposition[20] of $\{\hat{\Gamma}_0, \hat{\Gamma}_1\}$ and

$$\hat{S} = \begin{bmatrix} S_{11} & S_{12} & \mathbf{0} \\ \mathbf{0} & S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{T} = \begin{bmatrix} T_{11} & T_{12} & \mathbf{0} \\ \mathbf{0} & T_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Phi \end{bmatrix}, \qquad \hat{Z}^T = \begin{bmatrix} Z_1 & \mathbf{0} \\ Z_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \hat{Q} = \begin{bmatrix} Q_1 & \mathbf{0} \\ Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{46}$$

Note that in the expression above the auxiliary matrix Φ is in the lower (explosive) block because of our simplifying assumption that $m^*(\theta) = m$. When $m^*(\theta) < m$, part of the matrix Φ would belong in the stable block. As mentioned above, we made this simplifying assumption without loss of generality and only to simplify the exposition.

Second, the QZ decomposition chosen by the algorithm groups the explosive dynamics of the model in the second and third system of equations in (45), which are reported below as (47)

$$\begin{bmatrix} S_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \xi_2 \\ \omega_t \end{bmatrix} = \begin{bmatrix} T_{22} & \mathbf{0} \\ \mathbf{0} & \Phi \end{bmatrix} \begin{bmatrix} \xi_{2,t-1} \\ \omega_{t-1} \end{bmatrix} + \begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t\right). \tag{47}$$

In the third step, the following restrictions are imposed,

$$\underset{n\times 1}{\xi_{2,0}} = 0, \tag{48}$$

$$\underset{m\times 1}{\omega_0} = 0, \tag{49}$$

$$\begin{bmatrix} Q_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \left(\hat{\Psi}\hat{\varepsilon}_t + \hat{\Pi}\eta_t\right) = 0. \tag{50}$$

[20] Note that the inner matrices of $\{\hat{S}, \hat{T}, \hat{Z}^T, \hat{Q}\}$ are the same as those which define the matrices $\{S, T, Z^T, Q\}$ found from the QZ decomposition using the methodology of Lubik and Schorfheide (2003).

Recalling the definition of $\hat{\Psi}$ and $\hat{\Pi}$ in (44), then equation (50) can be written as

$$\underbrace{\begin{bmatrix} \widetilde{\Psi}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{p\times(\ell+m)} \underset{(\ell+m)\times 1}{\hat{\varepsilon}_t} + \underbrace{\begin{bmatrix} \widetilde{\Pi}_{n,2} & \widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix}}_{p\times p} \underset{p\times 1}{\eta_t} = 0, \tag{51}$$

where $\widetilde{\Psi} \equiv Q'\Psi$ and $\widetilde{\Pi} \equiv Q'\Pi$. Equation (51) shows transparently how the explosive auxiliary process that we append in our augmented representation helps to solve the original LRE model under indeterminacy. The system of equations in (51) is determinate, as the number of equations defined by the explosive roots of the system equals the number of expectational errors of the model. Thus, the necessary condition for the existence of a bounded solution for the augmented representation is satisfied. Assuming that the columns of the matrix associated with the vector of non-fundamental shocks, η_t, are linearly independent, we can impose linear restrictions on the forecast errors as a function of the augmented vector of exogenous shocks $\hat{\varepsilon}_t \equiv (\varepsilon_t, \nu_t)'$,

$$\eta_t = -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 & \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \hat{\varepsilon}_t.$$

More compactly,

$$\eta_t = C_1\varepsilon_t + C_2\nu_t, \tag{52}$$

where $C_1 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2 \\ \mathbf{0} \end{bmatrix}$ and $C_2 \equiv -\begin{bmatrix} \widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2} \\ -\mathbf{I} \end{bmatrix}$ are a function of the structural parameters of the model.

The last step of Sims' algorithm combines the restrictions in (48), (49) and (52) with the stationary block derived from the QZ decomposition in (45),

$$\begin{aligned} \xi_{1,t} &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}(\widetilde{\Psi}_1\varepsilon_t + \widetilde{\Pi}_1\eta_t) \\ &= S_{11}^{-1}T_{11}\xi_{1,t-1} + S_{11}^{-1}\left(\widetilde{\Psi}_1 + \widetilde{\Pi}_1 C_1\right)\varepsilon_t + S_{11}^{-1}\left(\widetilde{\Pi}_1 C_2\right)\nu_t. \end{aligned} \tag{53}$$

8.1.3 Mapping of normalization in Lubik and Schorfheide (2004) to Bianchi-Nicolò

We prove the equivalence between the parametrization of the Lubik-Schorfheide indeterminate equilibrium $\theta^{LS} \in \Theta^{LS}$ and the Bianchi-Nicolò equilibrium parametrized by $\theta^{BN} \in \Theta^{BN}$. In particular, we show that there is a unique mapping between the linear restrictions imposed in each of the two methodologies on the forecast errors to guarantee the existence of at least a bounded solution. As shown in Subsection 8.1.2, the method by Lubik and Schorfheide (2003) imposes the following restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks,

ε_t, and the sunspot shocks introduced in their specification, ζ_t,

$$\underset{p\times 1}{\eta_t} = \left(\underset{p\times n}{V_1}\underset{n\times \ell}{N} + \underset{p\times m}{V_2}\underset{\substack{m\times\ell\\ m\times\ell}}{\widetilde{M}}\right)\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{V_2}\underset{m\times 1}{\zeta_t}. \tag{54}$$

Using the methodology proposed in this paper, Subsection 8.1.2 shows that the restrictions on the non-fundamental shocks, η_t, as a function of the exogenous shocks, ε_t, and the sunspot shocks, v_t, are

$$\underset{p\times 1}{\eta_t} = \underset{p\times\ell}{C_1}\underset{\ell\times 1}{\varepsilon_t} + \underset{p\times m}{C_2}\underset{m\times 1}{\nu_t}, \tag{55}$$

where

$$C_1 \equiv -\begin{bmatrix}\widetilde{\Pi}_{n,2}^{-1}\widetilde{\Psi}_2\\ \mathbf{0}\end{bmatrix} \quad \text{and} \quad C_2 \equiv -\begin{bmatrix}\widetilde{\Pi}_{n,2}^{-1}\widetilde{\Pi}_{f,2}\\ -\mathbf{I}\end{bmatrix}.$$

Post-multiplying equation (54) and (55) by ε_t' and taking expectations on both sides,

$$\begin{aligned}
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times n}{V_1}\underset{n\times\ell}{N}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{V_2}\underset{m\times\ell}{\widetilde{M}}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}},\\
\underset{p\times l}{\Omega_{\eta\varepsilon}} &= \underset{p\times\ell}{C_1}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{p\times m}{C_2}\underset{m\times l}{\Omega_{\nu\varepsilon}}
\end{aligned}$$

Pre-multiplying by V_2' and equating the equations,

$$\underset{m\times\ell}{\widetilde{M}}\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} = \left(\underset{m\times p}{V_2'}\underset{p\times\ell}{C_1} - \underset{m\times p}{V_2'}\underset{p\times n}{V_1}\underset{n\times\ell}{N}\right)\underset{\ell\times l}{\Omega_{\varepsilon\varepsilon}} + \underset{m\times p}{V_2'}\underset{p\times m}{C_2}\underset{m\times l}{\Omega_{\nu\varepsilon}}. \tag{56}$$

Using the properties of the *vec* operator, the following result holds

$$\underset{(m\times\ell)\times 1}{vec(\widetilde{M})} = \underset{(m\times\ell)\times(m\times\ell)}{(\Omega_{\varepsilon\varepsilon}\otimes I_m)^{-1}}\left[\underset{(m\times\ell)\times\ell^2}{\left[I_l\otimes\left(V_2'C_1 - V_2'V_1N\right)\right]}\underset{\ell^2\times 1}{vec\left(\Omega_{\varepsilon\varepsilon}\right)} + \underset{(m\times\ell)\times(m\times\ell)}{\left(I_l\otimes V_2'C_2\right)}\underset{(m\times\ell)\times 1}{vec\left(\Omega_{\nu\varepsilon}\right)}\right]. \tag{57}$$

Equation (57) is the first relevant equation to show the mapping between the representation in Lubik and Schorfheide (2003) and our representation. For a given variance-covariance matrix of the exogenous shocks, $\Omega_{\varepsilon\varepsilon}$, that is common between the two representations, equation (57) tells us that the covariance structure, $\Omega_{\nu\varepsilon}$, of the sunspot shock in our representation with the exogenous shocks has a unique mapping to the matrix, $\widetilde{M}$, in Lubik and Schorfheide (2003). Clearly, equation (56) can also be used to derive the mapping from their representation to our method.

We now show how to derive the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Considering again equation (54) and (55), we post-multiply by ζ_t' and take expectations on both sides,

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{V_2}\ \underset{m\times m}{\Omega_{\zeta\zeta}}\,,\\ \underset{p\times m}{\Omega_{\eta\zeta}} &= \underset{p\times m}{C_2}\ \underset{m\times m}{\Omega_{\nu\zeta}}\end{aligned}$$

Pre-multiplying both equations by V_2' and equating them,

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \underset{m\times m}{\Omega_{\zeta\nu}}\ \underset{m\times m}{\left(V_2'C_2\right)'}. \tag{58}$$

Finally, to obtain an expression for $\Omega_{\zeta\nu}$, we post-multiply equation (54) and (55) by ν_t' and taking expectations

$$\begin{aligned}\underset{p\times m}{\Omega_{\eta\nu}} &= \left(\underset{p\times n}{V_1}\underset{n\times\ell}{N} + \underset{p\times m}{V_2}\ \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{V_2}\ \underset{m\times m}{\Omega_{\zeta\nu}}\,,\\ \underset{p\times m}{\Omega_{\eta\nu}} &= \underset{p\times\ell}{C_1}\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{p\times m}{C_2}\ \underset{m\times m}{\Omega_{\nu\nu}}\end{aligned}$$

Pre-multiplying both equations by V_2' and solving for $\Omega_{\zeta\nu}$,

$$\underset{m\times m}{\Omega_{\zeta\nu}} = \left(\underset{m\times p}{V_2'}\ \underset{p\times\ell}{C_1} - \underset{m\times p}{V_2'}\ \underset{p\times n}{V_1}\ \underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}} + \underset{m\times m}{\left(V_2'C_2\right)}\ \underset{m\times m}{\Omega_{\nu\nu}}\,. \tag{59}$$

Post-multiplying (59) by $\underset{m\times m}{(V_2'C_2)'}$ and using (58), then

$$\underset{m\times m}{\Omega_{\zeta\zeta}} = \left(\underset{m\times p}{V_2'}\ \underset{p\times\ell}{C_1} - \underset{m\times p}{V_2'}\ \underset{p\times n}{V_1}\ \underset{n\times\ell}{N} - \underset{m\times\ell}{\widetilde{M}}\right)\underset{\ell\times m}{\Omega_{\varepsilon\nu}}\underset{m\times m}{\left(V_2'C_2\right)'} + \underset{m\times m}{\left(V_2'C_2\right)}\ \underset{m\times m}{\Omega_{\nu\nu}}\ \underset{m\times m}{\left(V_2'C_2\right)'}. \tag{60}$$

Therefore, equation (60) defines the mapping between the variance-covariance matrix, $\Omega_{\nu\nu}$, of the sunspot shocks in our representation to the variance-covariance matrix, $\Omega_{\zeta\zeta}$, of the sunspot shocks in Lubik and Schorfheide (2003). Together with equation (57), we show that this equation defines the one-to-one mapping between the parametrization in Lubik and Schorfheide $\{\Theta, \Theta^{LS}\}$ and the parametrization in Bianchi-Nicolò $\{\Theta, \Theta^{BN}\}$.

8.2 Appendix B

Table 4: Posterior distribution of model parameters under 2-degree indeterminacy

	$\{\nu_1=\nu_\pi,\nu_2=\nu_y\}$		$\{\nu_1=\nu_\pi,\nu_2=\nu_b\}$		$\{\nu_1=\nu_y,\nu_2=\nu_b\}$	
	Mean	90% prob. int.	Mean	90% prob. int.	Mean	90% prob. int.
$100(\lambda_l^{-1}-1)$	0.026	[0.016,0.036]	0.025	[0.014,0.036]	0.028	[0.016,0.039]
κ	0.038	[0.030,0.046]	0.041	[0.033,0.049]	0.039	[0.031,0.047]
g	0.47	[0.42,0.53]	0.45	[0.41,0.48]	0.45	[0.41,0.49]
π^*	0.91	[0.48,1.46]	0.90	[0.42,1.35]	0.92	[0.43,1.40]
ϕ_π	0.32	[0.16,0.54]	0.30	[0.18,0.41]	0.24	[0.11,0.37]
ϕ_q	0.04	[0.02,0.08]	0.03	[0.01,0.04]	0.04	[0.02,0.06]
σ_q	1.51	[0.75,2.60]	1.60	[0.78,2.41]	2.57	[1.33,3.78]
σ_s	0.12	[0.10,0.15]	0.11	[0.09,0.13]	0.11	[0.09,0.13]
σ_i	0.13	[0.10,0.16]	0.12	[0.10,0.15]	0.12	[0.10,0.14]
ρ_q	0.80	[0.67,0.91]	0.81	[0.72,0.91]	0.74	[0.64,0.83]
ρ_s	0.88	[0.79,0.94]	0.88	[0.82,0.94]	0.87	[0.81,0.93]
σ_{ν_1}	0.28	[0.23,0.32]	0.30	[0.26,0.33]	0.73	[0.64,0.83]
σ_{ν_2}	0.71	[0.63,0.81]	6.58	[3.87,9.79]	5.59	[2.95,8.23]
$\varphi_{\nu_1,i}$	-0.59	[-0.70,-0.46]	-0.63	[-0.73,-0.52]	-0.21	[-0.41,-0.01]
$\varphi_{\nu_1,q}$	0.43	[0.18,0.68]	0.41	[0.23,0.58]	0.55	[0.37,0.75]
$\varphi_{\nu_1,s}$	0.62	[0.44,0.73]	0.60	[0.48,0.72]	-0.54	[-0.70,-0.40]
$\varphi_{\nu_2,i}$	-0.37	[-0.57,-0.15]	-0.71	[-0.82,-0.60]	-0.68	[-0.88,-0.49]
$\varphi_{\nu_2,q}$	0.22	[-0.15,0.57]	-0.32	[-0.59,-0.08]	-0.56	[-0.78,-0.34]
$\varphi_{\nu_2,s}$	-0.63	[-0.73,-0.49]	-0.55	[-0.70,-0.40]	-0.37	[-0.52,-0.21]
Marginal density	-83.3		-81.6		-80.3	

Table 5: Raftery-Lewis Diagnostics for each parameter chain in Galí (forthcoming)

Variable	Thin	Burn	Total (N)	Nmin	I-stat
$100(\lambda_l^{-1} - 1)$	1	2	1235	1286	0.960
κ	1	2	1372	1286	1.067
g	1	2	1316	1286	1.023
π^*	1	2	1262	1286	0.981
ϕ_π	1	2	1372	1286	1.067
ϕ_q	1	3	1432	1286	1.114
σ_q	1	2	1316	1286	1.023
σ_s	1	3	1185	1286	0.921
σ_i	1	3	1402	1286	1.090
ρ_q	1	4	1698	1286	1.320
ρ_s	1	3	1499	1286	1.166
σ_{ν_π}	1	3	1558	1286	1.212
σ_{ν_y}	1	2	1372	1286	1.067
$\varphi_{\nu_\pi,i}$	1	2	1343	1286	1.044
$\varphi_{\nu_\pi,q}$	1	4	1662	1286	1.292
$\varphi_{\nu_\pi,s}$	1	4	1627	1286	1.265
$\varphi_{\nu_y,i}$	1	2	1343	1286	1.044
$\varphi_{\nu_y,q}$	1	3	1494	1286	1.162
$\varphi_{\nu_y,s}$	1	2	1261	1286	0.981

The table reports the Raftery-Lewis diagnostics for each parameter chain. We consider the $5th$ quantile, $q = 0.05$, with an accuracy $r = 0.01$ and a probability $s = 0.9$ of obtaining an estimate in the interval $(q - r, q + r)$. The diagnostics reports the suggested number of burn-in iterations ("Burn"), the suggested number of iterations ("Total (N)"), the suggesed minimum number of iterations based on zero autocorrelation ("Nmin") and the dependence factor ("I-stat"). The dependence factor is computed as I-stat = (Burn + Total)/Nmin, and interpreted as the proportional increase in the number of iterations attributable to autocorrelation.

Table 6: Raftery-Lewis Diagnostics for each parameter chain in LS (2004)

	Mixture					**Random walk**				
	Thin	Burn	Total (N)	Nmin	I-stat	Thin	Burn	Total (N)	Nmin	I-stat
ψ_π	2	5	2981	1286	2.318	4	14	5790	1286	4.502
ψ_q	1	2	1299	1286	1.010	1	1	1287	1286	1.000
ρ_R	1	3	1545	1286	1.201	1	3	1432	1286	1.113
τ	1	3	1564	1286	1.216	1	3	1538	1286	1.196
κ	2	7	3037	1286	2.362	1	4	1654	1286	1.286
ρ_g	2	7	2967	1286	2.307	4	15	6287	1286	4.889
ρ_z	1	2	1332	1286	1.036	1	2	1321	1286	1.027
r^*	1	3	1468	1286	1.141	1	3	1474	1286	1.146
π^*	1	3	1500	1286	1.166	3	11	4136	1286	3.216
σ_R	1	2	1366	1286	1.062	1	3	1526	1286	1.187
σ_g	1	2	1372	1286	1.067	1	3	1384	1286	1.076
σ_z	1	2	1304	1286	1.014	1	1	1285	1286	0.999
ρ_{gz}	2	7	3051	1286	2.372	2	8	2972	1286	2.311
σ_η	2	6	2758	1286	2.145	1	3	1390	1286	1.081

The table reports the Raftery-Lewis diagnostics for each parameter chain using the hybrid ("Mixture") and the random walk algorithm ("Random walk"). We consider the $5th$ quantile, $q = 0.05$, with an accuracy $r = 0.01$ and a probability $s = 0.9$ of obtaining an estimate in the interval $(q - r, q + r)$. The diagnostics reports the suggested number of burn-in iterations ("Burn"), the suggested number of iterations ("Total (N)"), the suggesed minimum number of iterations based on zero autocorrelation ("Nmin") and the dependence factor ("I-stat"). The dependence factor is computed as I-stat = (Burn + Total)/Nmin, and interpreted as the proportional increase in the number of iterations attributable to autocorrelation.